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Coulomb's law, Electric field intensity & Flux density

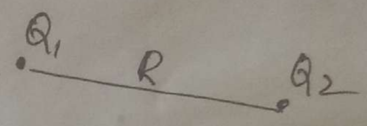
Coulomb's law :-

"Coulomb's law states that, the electrostatic force between any two point charges separated in a free space or vacuum is

i) directly proportional to the product of two charges &

ii) inversely proportional to the square of the distance between them.

i.e., $F \propto \frac{Q_1 Q_2}{R^2}$



$$\Rightarrow F = K \frac{Q_1 Q_2}{R^2} \text{ Newton.}$$

where,

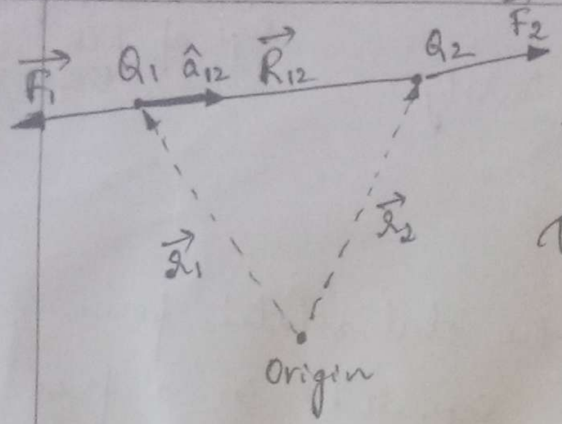
Q_1 & Q_2 are +ve or -ve charges in coulomb.

K is the proportionality constant.

$$K = \frac{1}{4\pi\epsilon_0} = 9\times10^9$$

& $\epsilon_0 = 8.854\times10^{-12}$ F/m is the permittivity of free space.

The direction of the force is always along the line joining the two charges. & is repulsive if the charges are alike in sign & attractive if they are of opposite signs.



Here, $\vec{r}_1$ & $\vec{r}_2$ are position vectors locating Q_1 & Q_2 respectively.

The vector force $\vec{F_2}$ on Q_2 due to Q_1 is given by,

$$\vec{F_2} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q_1 Q_2}{R_{12}^2}\,\hat{a}_{12}$$

Where, $\hat{a}_{12}$ is the unit vector in the direction of R_{12}

$$\hat{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}.$$

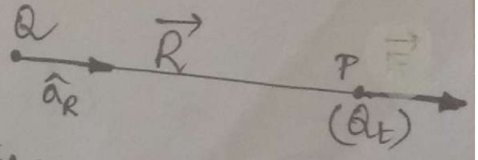
IIIly $\boxed{\vec{F_1} = -\vec{F_2}}$.

* <u>Electric field Intensity [EFI]</u> :—

"Electric field intensity at any point is defined as the vector force on a unit <u>positive charge</u> (test charge) placed at that point."

EFI is given by,

$$\vec{E} = \frac{\vec{F}}{Q_t}$$



The force experienced on a test charge, at point P.

$$\vec{F} = \frac{Q\,Q_t}{4\pi\epsilon_0 R^2}\,\hat{a}_R$$

or $$\frac{\vec{F}}{Q_t} = \frac{Q}{4\pi\epsilon_0 R^2}\,\hat{a}_R$$

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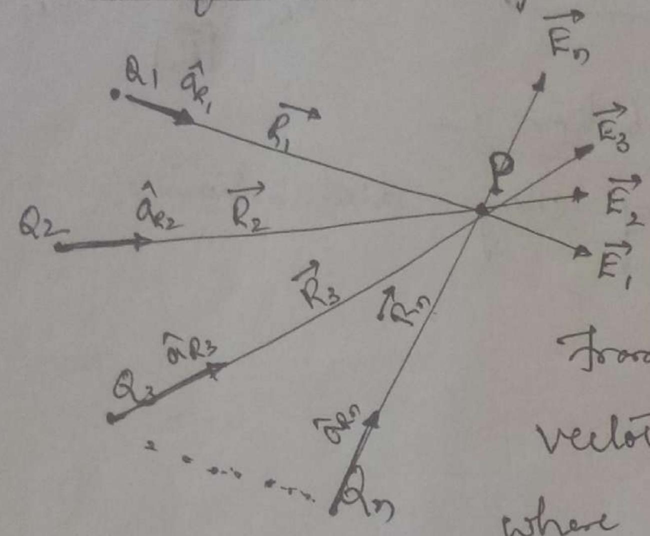
$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}\,\hat{a}_R$$

The above expression is the E.F.I due to a single point charge Q.

*** Electric field Intensity due to n point charges :-**



Consider 'n' point charges $Q_1, Q_2 \ldots Q_n$ in free space.

From the figure, $\vec{R_1}$ is the vector from Q_1 to point P where E.F.I is to be measured. Ily $\vec{R_2}$ is the vector from Q_2 & $\vec{R_n}$ is the vector from Q_n.

⇒ $\hat{a}_{R_1}, \hat{a}_{R_2} \ldots \hat{a}_{R_n}$ are unit vectors. The unit vector $\hat{a}_{R_1}$ indicates the direction of $\vec{E_1}$, $\hat{a}_{R_2}$ indicates the direction of $\vec{E_2}$ & $\hat{a}_{R_n}$ indicates the direction of $\vec{E_n}$.

The total EFI at point P is the vector sum of the intensities of individual field.

$$\vec{E}_{total} = \vec{E_1} + \vec{E_2} + \ldots + \vec{E_n} \quad \text{—— (1)}$$

where $\vec{E_1} = \dfrac{Q_1}{4\pi\epsilon_0 R_1^2}\hat{a}_{R_1}$, $\vec{E_2} = \dfrac{Q_2}{4\pi\epsilon_0 R_2^2}\hat{a}_{R_2} \ldots$

& $\vec{E_n} = \dfrac{Q_n}{4\pi\epsilon_0 R_n^2}\hat{a}_{R_n}$.

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$$\vec{E}_{total} = \frac{Q_1}{4\pi\epsilon_0 R_1^2}\hat{a}_{R_1} + \frac{Q_2}{4\pi\epsilon_0 R_2^2}\hat{a}_{R_2} + \cdots + \frac{Q_n}{4\pi\epsilon_0 R_n^2}\hat{a}_{R_n}$$

$$\boxed{\vec{E}_{total} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 R_m^2}\hat{a}_{R_m}}$$

where,

$m = 1, 2, 3 \ldots n$.

$\hat{a}_{R_1} = \frac{\vec{R}_1}{|\vec{R}_1|}$, $\hat{a}_{R_2} = \frac{\vec{R}_2}{|\vec{R}_2|} \ldots$

* **Types of charge distribution :-**

There are 4 types of charge distribution.

1) Point charge.
2) Line charge.
3) Surface charge &
4) Volume charge.

1) **Point charge distribution :-**

A point charge is a charge whose dimension is very very small compared to region surrounding it.

→ It has only position but no dimension.

→ A point charge can be positive or negative.

2) **Line charge distribution :-**

If the charge distribution is such that the point charges are distributed uniformly along a line, then it is referred as line charge distribution.

For a line charge, the line charge density ρ_L can be defined as charge per unit lengths.

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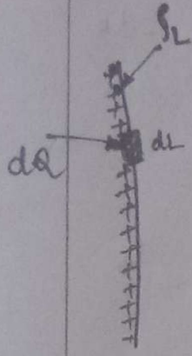
i.e., $\boxed{\rho_L = \dfrac{dQ}{dL}}$ The Unit of ρ_L is C/m.

$\Rightarrow dQ = \rho_L \, dL$ where dQ is the charge on differential length dL.



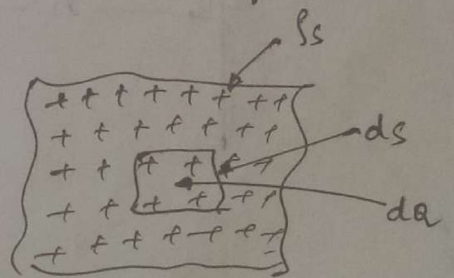
eg:- a wire of Uniform Cross section

3) <u>Surface charge distribution</u> :-

If the point charges are distributed uniformly over a two dimensional surface then it is called Surface charge distribution.

For a surface charge, the surface charge density is defined as the charge per Unit surface area.

i.e., $\boxed{\rho_S = \dfrac{dQ}{dS}}$



The Unit of ρ_S is C/m^2

$dQ = \rho_S \, dS$.

dQ → charge on differential surface dS

Ex:- The plates of a ||el plate capacitor.

4) <u>Volume charge distribution</u> :-

If the point charges are distributed continuously throughout a volume, then it is referred as volume charge distribution.

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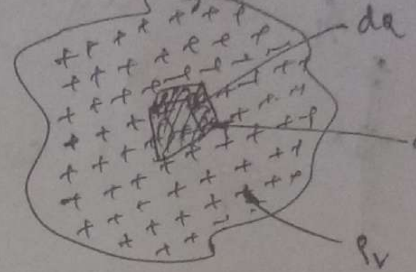
For a volume charge, the volume charge density (ρ_v) is defined as the charge per unit volume.

i.e., $$\rho_v = \frac{dQ}{dv}$$

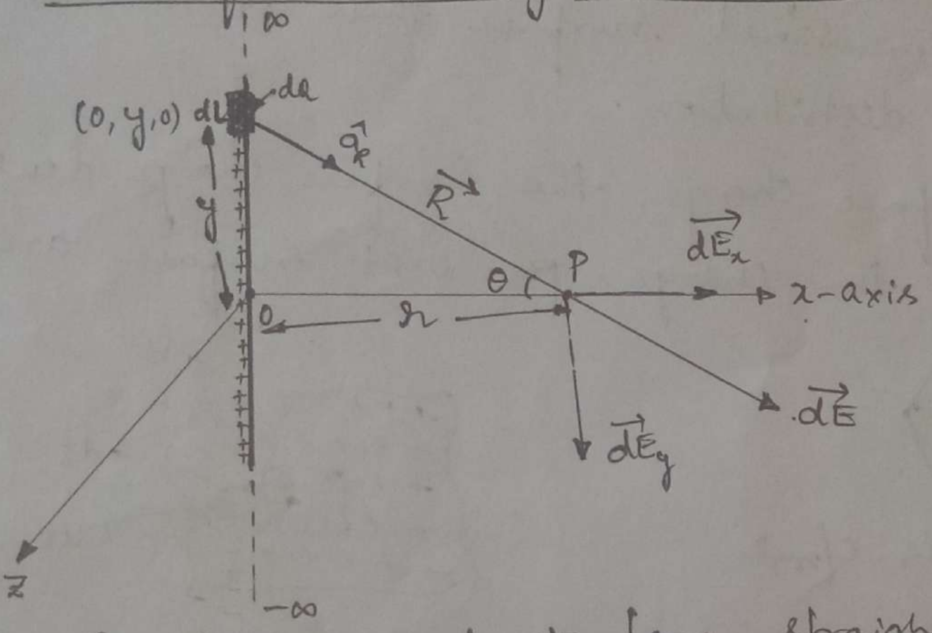


The unit of ρ_v is C/m^3

$$dQ = \rho_v \, dv$$

dQ → charge on differential volume dv.

Electric field intensity due to line charge :-



Consider an infinite long straight line charge of line charge density ρ_L. Let this line charge lies along y axis from $-\infty$ to $+\infty$.

Let P be a point on x-axis at which EFI is to be measured.

The co-ordinates of dQ are (0, y, 0) & the co-ordinates of point P are (r, 0, 0).

The E.F.I at P due to dQ is given by

$$\vec{dE} = \frac{dQ}{4\pi\epsilon_0 R^2}\,\hat{a}_R \quad \text{—— ①}$$

where, $\hat{a}_R = \dfrac{\vec{R}}{|\vec{R}|}$, $\vec{R} = r\,\hat{a}_x - y\,\hat{a}_y$ & $|\vec{R}| = \sqrt{r^2+y^2}$

$$\hat{a}_R = \frac{r\,\hat{a}_x - y\,\hat{a}_y}{\sqrt{r^2+y^2}}.$$

The line charge density $\rho_L = \dfrac{dQ}{dL}$ $\Rightarrow$ $dQ = \rho_L\, dL$

Since line is on y-axis

$$\boxed{dL = dy}$$

$$\therefore \; dQ = \rho_L\, dy.$$

Eqn ① becomes,

$$\vec{dE} = \frac{\rho_L\, dy}{4\pi\epsilon_0 (r^2+y^2)}\left[\frac{r\,\hat{a}_x - y\,\hat{a}_y}{\sqrt{r^2+y^2}}\right]$$

From the Symmetry, by eliminating y-component.

$$\vec{dE} = \frac{\rho_L\, dy}{4\pi\epsilon_0 (r^2+y^2)^{3/2}}\; r\,\hat{a}_x$$

Integrating $\vec{dE}$ from $-\infty$ to ∞, we get $\vec{E}$

$$\vec{E} = \int_{-\infty}^{\infty} dE = \int_{-\infty}^{\infty} \frac{\rho_L\, r\, dy}{4\pi\epsilon_0 (r^2+y^2)^{3/2}}\;\hat{a}_x.$$

From the figure $\tan\theta = y/r$

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$$y = r\tan\theta$$

$$dy = r\sec^2\theta \, d\theta.$$

If $y = -\infty$, $\theta = \tan^{-1}(-\infty) = -\pi/2$

$y = \infty$, $\theta = \tan^{-1}(\infty) = \pi/2$.

$$\vec{E} = \int_{\theta=-\pi/2}^{\pi/2} \frac{\rho_L \, r . r\sec^2\theta \, d\theta}{4\pi\epsilon_0 \left[r^2 + r^2\tan^2\theta\right]^{3/2}}$$

$$\left[r^2(1+\tan^2\theta)\right]^{3/2}$$

$$r^3(1+\tan^2\theta)^{3/2}$$

$$1+\tan^2\theta = \sec^2\theta$$

$$r^3(\sec^2\theta)^{3/2}$$

$$r^3\sec^3\theta$$

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{r^2\sec^2\theta \, d\theta}{r^3(1+\tan^2\theta)^{3/2}}$$

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{r\sec\theta}$$

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0 r} \int_{-\pi/2}^{\pi/2} \cos\theta . d\theta$$

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0 r}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = \frac{\rho_L}{4\pi\epsilon_0 r}\left[1-(-1)\right]$$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 r}\hat{a}_x} \quad V/m$$

In general $\vec{E} = \dfrac{\rho_L}{2\pi\epsilon_0 r}\hat{a}_r$

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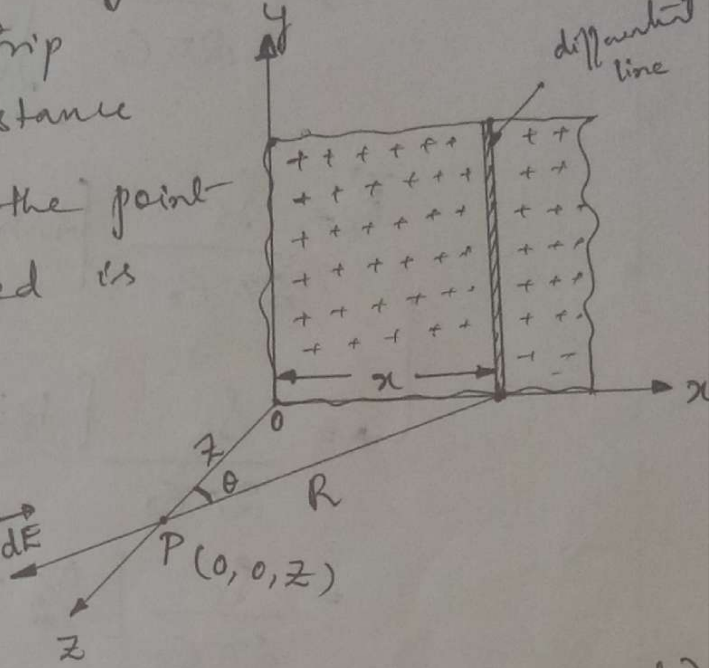
** Electric field due to sheet of charge :-

Consider a sheet of charge placed on xy plane. Considering symmetry, there is no field along x & y direction. Hence only Z component is present.



Consider a differential strip along y-direction. The distance from this line charge to the point P where EFI is desired is

$R = \sqrt{x^2 + z^2}$.

The E.F.I at P due to differential strip is

$$\vec{dE_z} = \frac{\rho_L}{2\pi\epsilon_0 R}\cos\theta \quad \text{—①} \quad \text{(only } z \text{ component is present)}$$

$$\rho_s = \frac{dQ}{ds} = \frac{\overbrace{dQ}^{\rho_L}}{dx\,dy}$$

$$\rho_s = \frac{\rho_L}{dx}$$

$$\rightarrow \rho_L = \rho_s\,dx$$

But $R = \sqrt{x^2 + z^2}$

$$\rho_L = \rho_s\,dx$$

From the fig $\cos\theta = \dfrac{z}{R} = \dfrac{z}{\sqrt{x^2+z^2}}$

Substituting eqn ① becomes,

$$\vec{dE_z} = \frac{\rho_s\,dx}{2\pi\epsilon_0\sqrt{x^2+z^2}} \times \frac{z}{\sqrt{x^2+z^2}}\,\hat{a}_z$$

$$\vec{dE_z} = \frac{\rho_s\,z\,dx}{2\pi\epsilon_0\,(x^2+z^2)}\,\hat{a}_z$$

Total E.F.I due to the sheet is,

$$\vec{E}_z = \int_{-\infty}^{\infty} d\vec{E}_z = \int_{x=-\infty}^{\infty} \frac{\rho_s z\, dx}{2\pi\epsilon_0 (x^2+z^2)} \hat{a}_z$$

$$\vec{E}_z = \frac{\rho_s}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{z}{(x^2+z^2)} dx \cdot \hat{a}_z$$

$$\vec{E}_z = \frac{\rho_s}{2\pi\epsilon_0} \left[\tan^{-1}\left(\frac{x}{z}\right)\right]_{x=-\infty}^{\infty}$$

$$= \frac{\rho_s}{2\pi\epsilon_0} \left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right]$$

$$= \frac{\rho_s}{2\pi\epsilon_0} \left[\pi/2 - (-\pi/2)\right] = \frac{\rho_s}{2\cancel{\pi}\epsilon_0} [\cancel{\pi}]$$

$$\boxed{\vec{E}_z = \frac{\rho_s}{2\epsilon_0} \hat{a}_z}$$

In general, $\vec{E} = \frac{\rho_s}{2\epsilon_0} \hat{a}_N$ where $\hat{a}_N$ is the unit vector normal to the sheet.

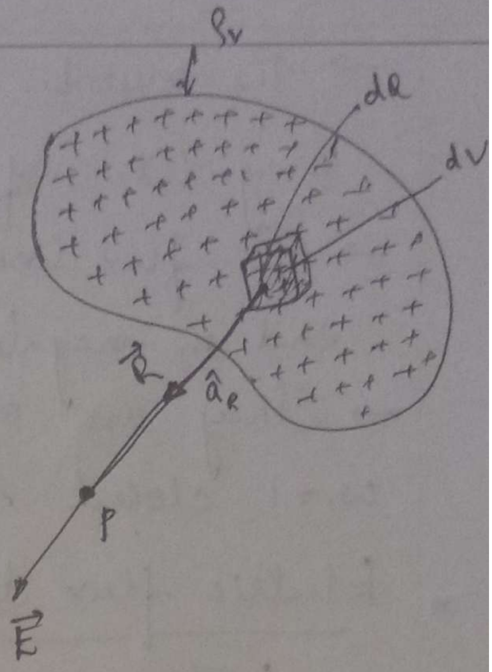
* E.F.I due to a Continuous volume charge distribution

For a volume charge distribution, the volume charge density ρ_v is given by

$$\rho_v = \frac{dQ}{dv}$$

$\Rightarrow dQ = \rho_v\, dv$ is the charge on differential volume dv.

The E.F.I at point P due to Point charge dQ is



$$\vec{dE} = \frac{dQ}{4\pi\epsilon_0 R^2}\hat{a}_R$$

The total field at P due to entire volume charge distribution is

$$\vec{E} = \int_{vol} \vec{dE} = \int_{vol} \frac{dQ}{4\pi\epsilon_0 R^2}\hat{a}_R$$

But $dQ = \rho_v \, dV$

$$\therefore \vec{E} = \int_{vol} \frac{\rho_v \, dV}{4\pi\epsilon_0 R^2}\hat{a}_R$$

$$\boxed{\vec{E} = \frac{\rho_v}{4\pi\epsilon_0}\int_{Vol}\frac{dV}{R^2}\hat{a}_R}$$

It is possible to perform the integration only when the shape is symmetrical.

* Electric flux (ψ):-

According to Faraday's, the electric field is visualized in terms of electric flux lines.

→ These are $\parallel^{el}$ lines of force & never intersect each other.

→ They are independent of the medium in which charges are kept.

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→ The number of lines of force is equal to the magnitude of the charge. → $\boxed{\psi = Q}$

→ The flux lines start from positive charges and end on negative charges.

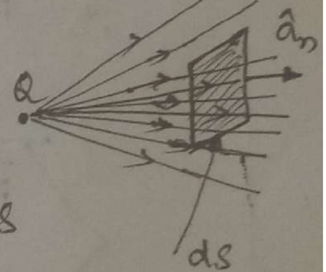
→ They are open lines & hence they are measured w.r.t closed surface.

* Electric flux density [$\vec{D}$] :-

" The net flux passing normal through the unit surface area is called the electric flux density. "

The SI unit of electric flux density is C/m^2

$$\boxed{\vec{D} = \frac{d\psi}{ds}\,\hat{a}_N}$$



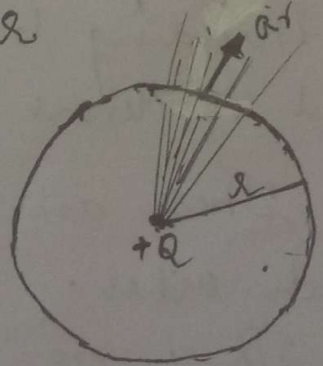
$d\psi$ → flux passing through the area ds

$\hat{a}_N$ → Unit vector normal to the surface area ds.

* $\vec{D}$ due to a point charge :-

Consider a point charge Q at the centre of an imaginary sphere of radius r

The flux lines emerging from the point charge are directed radially outwards.



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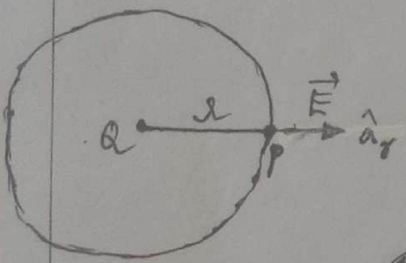
If The total flux through the sphere is Ψ & total area of the sphere is $4\pi r^2$, then

$$|\vec{D}| = \frac{\text{total flux}}{\text{total surface area}}$$

$$\vec{D} = \frac{\Psi}{4\pi r^2}\,\hat{a}_r \qquad \text{But } \Psi = Q$$

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2}\,\hat{a}_r}$$

* <u>Relationship between $\vec{D}$ & $\vec{E}$:-</u>



E.F.I at P (i.e, on the surface),

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{a}_r \quad \text{——} ①$$

The flux density at any point on the spherical surface,

$$\vec{D} = \frac{Q}{4\pi r^2}\,\hat{a}_r \quad \text{——} ②$$

from ① & ②

$$\vec{E} = \frac{\vec{D}}{\epsilon_0}$$

or $\boxed{\vec{D} = \epsilon_0\vec{E}}$

* Electric flux density due to line charge is

$$\boxed{\vec{D} = \frac{\rho_L}{2\pi r}\,\hat{a}_r}$$

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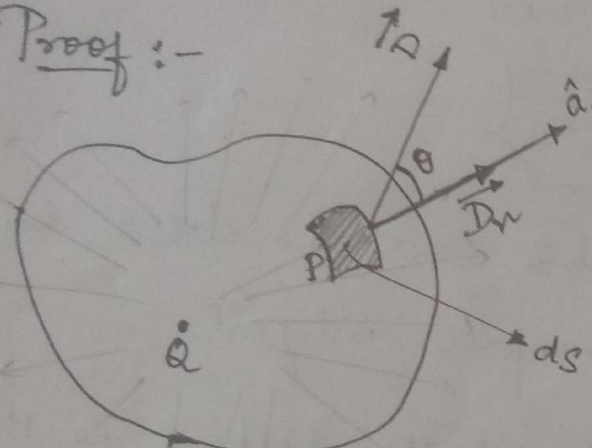
Module-2 Gauss's Law and Divergence

* Gauss's law:-

Statement - "The electric flux passing through any closed surface is equal to the total charge enclosed by that surface."

i.e., $$\boxed{\Psi = Q_{enclosed}}$$

Proof:-



Closed irregular surface

Let Q be the total charge enclosed by the irregular closed surface.

Consider a small differential surface ds at point P. As the surface is irregular, the direction of $\vec{D}$ is going to change from point to point on the surface.

The flux density $\vec{D}$ makes an angle θ with the normal direction at point P.

$$\vec{D}_n = \frac{d\psi}{ds}\, \hat{a}_N$$

$$d\psi = D_n\, ds$$

where $d\psi$ is the flux lines passing normal through the area ds.

From the fig,

$$\vec{D_n} = |\vec{D}| \cos\theta$$

where $\vec{D_n} \rightarrow$ flux density normal to the surface ds

$$\therefore \; d\psi = |\vec{D}| \cos\theta \, ds$$

$$\rightarrow d\psi = \vec{D} \cdot \vec{ds}$$

$|\vec{D}| = D$

$|\vec{ds}| = ds$

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$

$\vec{D} \cdot \vec{ds} = |\vec{D}| \, |\vec{ds}| \cos\theta$

The total flux passing through the entire closed surface is

$$\boxed{\psi = \int d\psi = \oint \vec{D} \cdot \vec{ds}}$$

here, $\oint \rightarrow$ indicates the integration over the closed surface & is called closed surface integral.

Irrespective of the shape of the surface & the charge distribution, total flux passing through the surface is equal to the total charge enclosed by that surface.

$$\boxed{\psi = \oint \vec{D} \cdot \vec{ds} = Q_{enclosed}}$$

i) If the charge distribution is of type line of density ρ_L, then

$$\psi = Q = \int \rho_L \, dL .$$

ii) If the charge distribution is of type surface of density ρ_s then $\psi = Q = \int \rho_s \, ds$.

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iii) If the charge distribution is of type volume of density ρ_v then
$$\psi = Q = \int \rho_v \, dv.$$
Thus Gauss's law can be applicable for any type of charge distributions.

* Special Gaussian Surfaces :-

1) The surface must be closed.
2) The surface may be irregular but should be sufficiently large so as to enclose the entire charge.
3) At each point on the surface $\vec{D}$ is normal to the surface.

* Applications of Gauss's law :-

1) Application of Gauss's law to Spherical Co-ordinate System (SCS) -

Consider a point charge Q at the origin of a SCS as shown in figure.

To apply Gauss's law, consider a spherical surface around Q. This spherical surface is a Gaussian surface & it satisfies required condition.

The flux density $\vec{D}$ is always directed radially outwards along $\hat{a}_r$ & is normal to the spherical surface.

Now, $\vec{D}$ due to a point charge, $\vec{D} = \dfrac{Q}{4\pi r^2} \hat{a}_r$ —①

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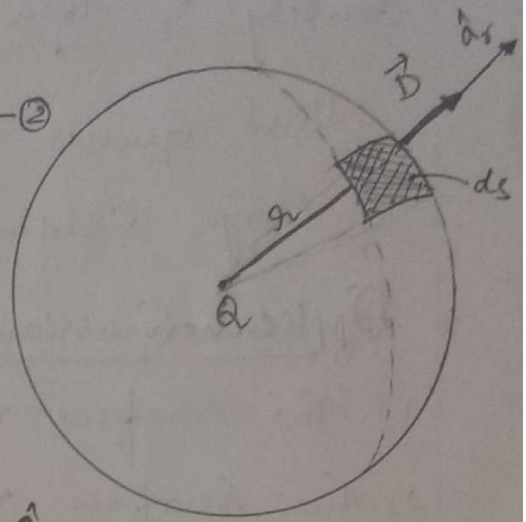
Consider a differential surface area ds on a spherical surface.

i.e., $\vec{ds} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$ —— ②



From Gauss's law

$$\psi = \oint \vec{D} \cdot \vec{ds} \quad \text{—— ③}$$

Consider RHS, from ① & ②

$$\oint \vec{D} \cdot \vec{ds} = \oint \frac{Q}{4\pi r^2} \hat{a}_r \cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

$$= \oint \frac{Q}{4\pi \cancel{r^2}} \cancel{r^2} \sin\theta \, d\theta \, d\phi \qquad \because \hat{a}_r \cdot \hat{a}_r = 1$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{Q}{4\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{Q}{4\pi} [-\cos\theta]_0^{\pi} \, [\phi]_0^{2\pi}$$

$$= \frac{Q}{\cancel{4\pi}} \times \cancel{2} \times \cancel{2\pi} = Q$$

$\therefore \boxed{\psi = Q}$ ..... Gauss's law is proved.

2> Application of Gauss's law to Cylindrical co-ordinate System [CCS]

* Derive an expression for EFI due to an Uniformly charged line of density ρ_L using Gauss's law.

Consider an uniformly charged line of density ρ_L lying along Z-axis.

Consider a gaussian surface as the right circular cylinder with Z-axis as its axis & radius r. The length of the cylinder is L.

The flux density at any point on the surface is directed radially outwards i.e., in the $\hat{a}_r$ direction.

Consider a differential surface area ds on the cylindrical surface. This area is normal to $\hat{a}_r$ direction.

Applying Gauss's law.

$$\psi = Q = \oint \vec{D} \cdot \vec{ds} \quad \text{—— (1)}$$

$|\vec{D}| = D$

But $\vec{D} = |\vec{D}|\, \hat{a}_r$ (only radial component —— (2) is present)

For a CCS,

$$\vec{ds} = r\, d\phi\, dz\, \hat{a}_r \quad \text{—— (3)}$$ (area normal to $\hat{a}_r$ direction)

$\therefore$ eqn (1) becomes.

$$Q = \oint D\, \hat{a}_r \cdot r\, d\phi\, dz\, \hat{a}_r \quad [\text{but } \hat{a}_r \cdot \hat{a}_r = 1]$$

$$Q = \int_{\phi=0}^{2\pi} \int_{z=0}^{L} D\, r\, d\phi\, dz$$

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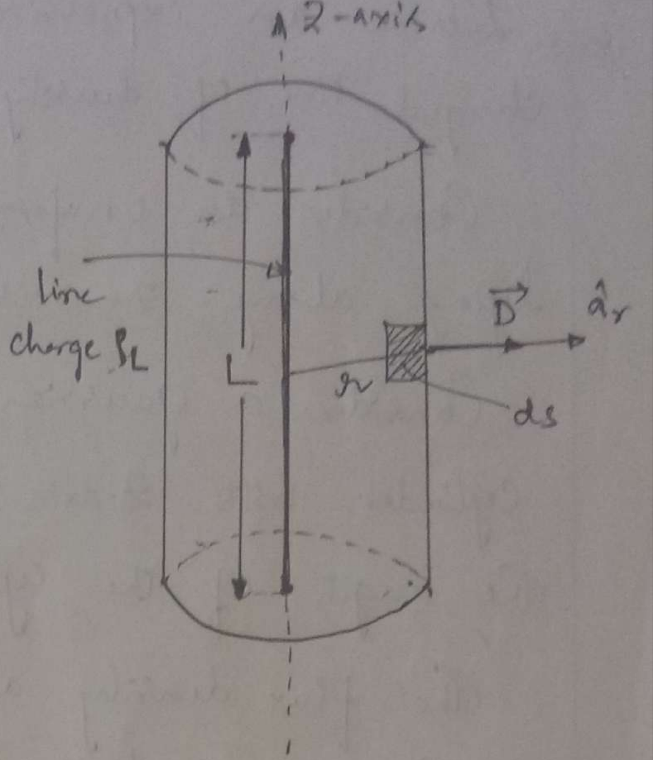
$$Q = D\, r \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{L} dz$$

$$Q = D\, r\, [\phi]_0^{2\pi}\, [z]_0^{L}$$

$$Q = D\, r\, 2\pi L$$

$$\therefore\ D = \frac{Q}{2\pi r L}$$



from eqn (2)

$$\vec{D} = \frac{Q}{2\pi r L}\, \hat{a}_r .$$

But the charge density $\rho_L = \frac{Q}{L}$

$$\therefore\ \boxed{\vec{D} = \frac{\rho_L}{2\pi r}\, \hat{a}_r}\ C/m^2$$

But WKT $\vec{D} = \epsilon_0 \vec{E}$

$$\Rightarrow \boxed{\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\rho_L}{2\pi \epsilon_0 r}\, \hat{a}_r}\ /m .$$

3) Gauss's law applied to differential volume element :-

Consider a closed Gaussian differential surface in the form of rectangular box. The sides of this element is Δx, Δy & Δz.

The value of $\vec{D}$ at the point P may be expressed as $\vec{D} = D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z$

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Applying Gauss's law,

$$Q = \oint \vec{D} \cdot \vec{ds} \quad \text{——} ①$$

The total surface integral is to be evaluated over six surfaces.

$$\therefore \oint \vec{D} \cdot \vec{ds} = \left\{ \int_{front} + \int_{back} + \int_{top} + \int_{bottom} + \int_{left} + \int_{right} \right\} \vec{D} \cdot \vec{ds}.$$

Consider the front surface.

$$\int_{front} \vec{D} \cdot \vec{ds} = \overrightarrow{D_{z\,front}} \cdot \vec{ds}_{front}$$

Using Taylor's series,

$$\overrightarrow{D_{z\,front}} = D_z + \frac{\Delta z}{2} \times \text{rate of change of } D_z \text{ wrt } z$$

$$= \left(D_z + \frac{\Delta z}{2} \frac{\partial D_z}{\partial z} \right) \hat{a}_z$$

$$\vec{ds}_{front} = \Delta x \, \Delta y \, \hat{a}_z$$

$$\therefore \int_{front} \vec{D} \cdot \vec{ds} = \left(D_z + \frac{\Delta z}{2} \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \quad \left(\because \hat{a}_z \cdot \hat{a}_z = 1 \right) \quad \text{——} ②$$

Illy

$$\int_{back} \vec{D} \cdot \vec{ds} = \overrightarrow{D_{z\,back}} \cdot \vec{ds}_{back}$$

$$\overrightarrow{D_{z\,back}} = \left(D_z - \frac{\Delta z}{2} \frac{\partial D_z}{\partial z} \right) \hat{a}_z$$

$$\vec{ds}_{back} = (\Delta x \, \Delta y) - \hat{a}_z$$

$$\int_{back} \vec{D} \cdot \vec{ds} = \left(D_z - \frac{\Delta z}{2} \frac{\partial D_z}{\partial z} \right) (-\Delta x \, \Delta y) \quad \text{——} ③$$

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Adding ② & ③

$$\int_{front} + \int_{back} = \frac{\partial D_z}{\partial z} \Delta x \Delta y \Delta z \quad \text{—— ④}$$

llly

$$\int_{left} + \int_{right} = \frac{\partial D_x}{\partial x} \Delta x \Delta y \Delta z \quad \text{—— ⑤}$$

$$\int_{top} + \int_{bottom} = \frac{\partial D_y}{\partial y} \Delta x \Delta y \Delta z \quad \text{—— ⑥}$$

$$\therefore \oint \vec{D} \cdot \vec{ds} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

But $\Delta x \Delta y \Delta z = \Delta V$

$$\therefore \oint \vec{D} \cdot \vec{ds} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta V$$

* **Divergence :-**

Imp "The divergence of the electric flux density $\vec{D}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero."

$$\boxed{\text{Divergence of } \vec{D} = \text{div } \vec{D} = \lim_{\Delta V \to 0} \frac{\oint \vec{D} \cdot \vec{ds}}{\Delta V}}$$

A positive divergence for any vector quantity indicates a source at that point. A negative divergence (convergence) indicates a sink at that

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The equation $\nabla \cdot \vec{D} = \rho_v$ is called Maxwell's 1st equation. It is also called point form of Gauss's law or Gauss's law in differential form.

** Gauss's - Divergence theorem :-

Statement - " The integral of the normal component of the flux density over any closed surface in an electric field is equal to the volume integral of the divergence of the flux density."

i.e, $$\oint \vec{D} \cdot \vec{ds} = \int_V (\nabla \cdot \vec{D})\, dV$$

Proof :- According to Gauss's law,

$$Q = \oint \vec{D} \cdot \vec{ds} \quad \text{——— ①}$$

expressing gauss- law per unit volume basis

$$\frac{Q}{\Delta V} = \frac{\oint \vec{D} \cdot \vec{ds}}{\Delta V}$$

Taking Lim $\Delta V \rightarrow 0$ i.e, volume shrinks to zero.

$$\lim_{\Delta V \rightarrow 0} \frac{Q}{\Delta V} = \lim_{\Delta V \rightarrow 0} \frac{\oint \vec{D} \cdot \vec{ds}}{\Delta V} \quad \text{——— ②}$$

But $\rho_v = \lim_{\Delta V \rightarrow 0} \frac{Q}{\Delta V}$ is the volume charge density

——— ③

Also $\quad \text{div}\,\vec{D} = \lim_{\Delta v \to 0} \frac{\oint \vec{D}\cdot\vec{ds}}{\Delta v}$ — ④

from ③ & ④, eqn ② becomes.

$$\rho_v = \text{div}\,\vec{D}$$

or $\quad \text{div}\,\vec{D} = \rho_v \Rightarrow \boxed{\nabla\cdot\vec{D} = \rho_v}$ — ⑤

Consider a gaussian surface, let Q be the charge within the volume,

$$Q = \int_v \rho_v \, dv \quad — ⑥$$

Using eqn ⑤ in ⑥

$$Q = \int_v (\nabla\cdot\vec{D})\, dv \quad — ⑦$$

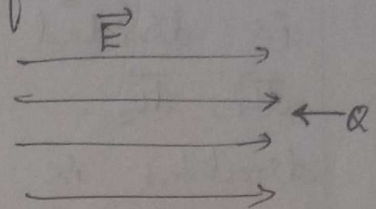
from eqn ① & ⑦

$$\boxed{Q = \oint_s \vec{D}\cdot\vec{ds} = \int_v (\nabla\cdot\vec{D})\, dv}$$

It converts surface integral into volume integral & vice versa. $\oint_s \rightleftharpoons \int_{vol}$

Module-2 - Energy and Potential
[Contd]

Consider an uniform electric field $\vec{E}$, The amount of force responsible to move the charge Q through a distance dl in the direction of the field is



$$\vec{F} = Q\vec{E} \quad \text{—— ①}$$

If we attempt to move the charge Q against the electric field, we have to exert a force equal & opposite to that exerted by the field. Thus work is said to be done.

i.e., $\vec{F}_{applied} = -Q\vec{E}$ —— ②

If a charge is moved through a differential distance $\vec{dL}$, against the direction of field $\vec{E}$, then the differential work done dW is given by,

$$\boxed{dW = -Q\vec{E}\cdot\vec{dL}}$$

(∵ $\vec{F} = Q\vec{E}$)
Work done = Force × distance

Thus, if a charge is moved from initial position to the final position, against the direction of $\vec{E}$, then the total work

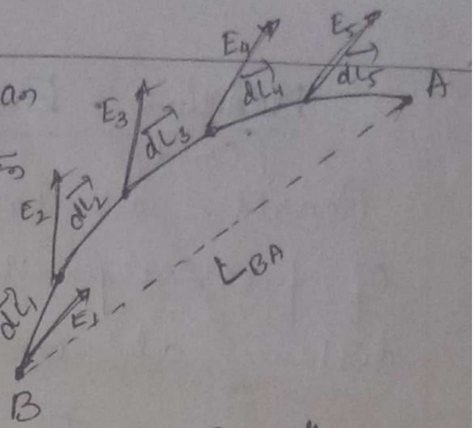
$$\boxed{W = -Q\int_{initial}^{final}\vec{E}\cdot\vec{dL}}$$ joules.

The Line Integral:

The line integral tells us to choose any arbitrary path, break it up into a large no. of very small segments, multiply the component of the field along each segment by the length of the segment & then add the result.

The procedure is indicated in figure.

The path has been chosen from an initial position B to a final position A. A Uniform Electric field is selected for simplicity. The path is divided into 5 segments.



$\vec{dL_1}, \vec{dL_2} \ldots \vec{dL_5}$, while the electric field in these directions is $\vec{E_1}, \vec{E_2} \ldots \vec{E_5}$

The work involved in moving a charge Q from B to A is given by,

$$W = -Q\left(\vec{E_1}\cdot\vec{dL_1} + \vec{E_2}\cdot\vec{dL_2} + \ldots \vec{E_5}\cdot\vec{dL_5}\right)$$

Since uniform field.

$$\vec{E_1} = \vec{E_2} = \ldots \vec{E_5}$$

$$\therefore \quad W = -Q\vec{E}\cdot\left(\vec{dL_1} + \vec{dL_2} + \ldots + \vec{dL_5}\right)$$

By vector law of addition,

$$\vec{dL_1} + \vec{dL_2} + \ldots + \vec{dL_5} = \vec{L_{BA}} \longrightarrow$$ vector directed from initial pt B to the final pt A

$$\therefore \quad W = -Q\vec{E}\cdot\vec{L_{BA}}$$

If the segments are infinitely small, then

$$W = -Q\int_B^A \vec{E}\cdot\vec{dL}$$

The work done depends on Q, $\vec{E}$ & $\vec{L_{BA}}$ & does not depend on the path joining B to A. i.e, it is <u>independent of the path selected</u>.

It depends only on the initial pt & final pt. i.e, $\vec{L_{BA}}$.

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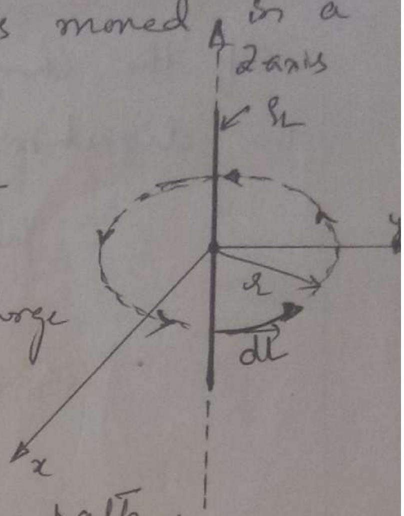
* Work done due to line charge distribution of density ρ_L

→ Work done is zero if a charge Q is moved in a circular path (along ϕ direction).



Consider a line charge of density ρ_L aligned along z-axis.

The field $\vec{E}$ due to infinite line charge is,

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 r}\,\hat{a}_r \quad \text{——} (1)$$

Since charge is moved along circular path,

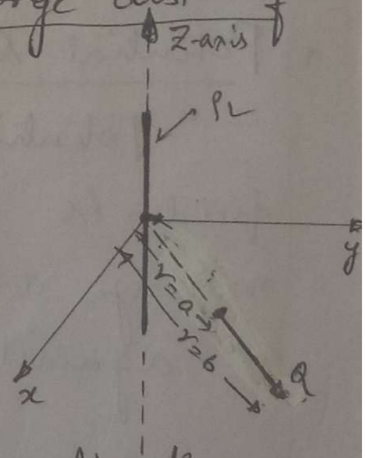
$$\vec{dL} = r\,d\phi\,\hat{a}_\phi \quad \text{——} (2)$$

W.K.T, $W = -Q\int_{initial}^{final} \vec{E}\cdot\vec{dL} = -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\epsilon_0 r}\,\hat{a}_r \cdot r\,d\phi\,\hat{a}_\phi$

$$\Rightarrow \boxed{W = 0} \quad (\because \hat{a}_r\cdot\hat{a}_\phi = 0)$$

∴ Work can be done only in radial direction

* Expression for work done due to line charge distn of density ρ_L in radial direction



Consider a line charge of density ρ_L as shown in figure.

The movement of the point charge Q is along radial direction & hence $\vec{dL}$ has no component in $\hat{a}_\phi$ & $\hat{a}_z$ direction.

i.e, $\vec{dL} = dr\,\hat{a}_r \quad \text{——} (1)$

The E.F due to infinite line charge

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 r}\,\hat{a}_r \quad \text{——} (2)$$

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W.K.T

$$W = -Q \int_{initial}^{final} \vec{E} \cdot \vec{dL}$$

If the charge is moved from $r=a$ to $r=b$ along the direction of $\vec{E}$.

$$W = -Q \int_{a}^{b} \frac{\rho_L}{2\pi\epsilon_0 r} dr \qquad (\hat{a}_r \cdot \hat{a}_r = 1)$$

$$W = -\frac{Q\rho_L}{2\pi\epsilon_0} [\ln r]_a^b$$

$$\Rightarrow \boxed{W = -\frac{Q\rho_L}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)}$$ as $b>a$. $\ln(b/a)$ is positive & work done is negative.

Suppose if Q is moved from b to a against the direction of $\vec{E}$

$$\boxed{W = \frac{Q\rho_L}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)}$$ work done is positive.

* Potential Difference :-

"Potential difference between two points in an electric field is defined as the amount of work done in moving a unit +ve charge from one point to another or against the field."

i.e., $V = -\int_{initial}^{final} \vec{E} \cdot \vec{dL}$

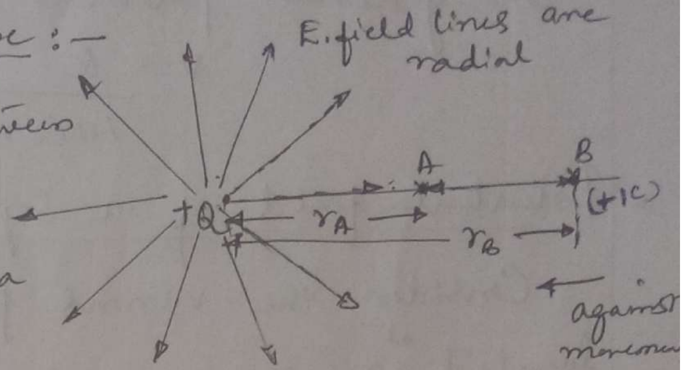
The p.d between two points A & B

$$V_{AB} = -\int_{B}^{A} \vec{E} \cdot \vec{dL}$$

V_{AB} is positive if the work is done by the external source is moving the unit +ve charge from B to A against the direction of E.

* The unit of P.d is J/C or volt (V).

* Potential due to point charge :-



The potential difference between points A & B at radial distances r_A & r_B from a point charge Q,

$$V_{AB} = -\int_B^A \vec{E} \cdot \vec{dL}$$

E. field due to point charge

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r \quad \text{——} (1)$$

& $\vec{dL} = dr\ \hat{a}_r$ —— (2) [∵ moving +1C charge against to field in radial direction]

$$\therefore V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr \quad (\because \hat{a}_r \cdot \hat{a}_r = 1)$$

$$\boxed{V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]}$$ as $r_B > r_A$, V_{AB} is +ve

* Absolute potential :- [Scalar potential]

"The absolute potential at any point in an electric field is defined as the work done in moving a unit +ve charge from infinity (reference pt at which potential is zero) to that point, against the direction of the field."

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The absolute potential at A is

$$\boxed{V_A = \frac{Q}{4\pi\epsilon_0 r_A}}$$

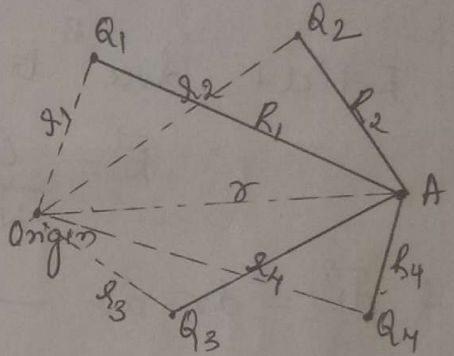
In general potential @ any point,

$$V = \frac{Q}{4\pi\epsilon_0 r}$$, potential is a Scalar quantity

* Potential field of a System of charges :-

Consider the various point charges $Q_1, Q_2, Q_3 \ldots Q_n$ located at $r_1, r_2, r_3 \ldots r_n$ respectively from the origin as shown.

Let A be the point where potential is to be measured due to all the charges



The potential at A due to Q_1 is

$$V_1 = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} = \frac{Q_1}{4\pi\epsilon_0 R_1}$$

111y potential at A due to Q_2 is

$$V_2 = \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} = \frac{Q_2}{4\pi\epsilon_0 R_2}$$

The net potential @ A due to all point charges

$$V_A = V_1 + V_2 + V_3 + \cdots + V_n$$

$$\Rightarrow = \frac{Q_1}{4\pi\epsilon_0 R_1} + \frac{Q_2}{4\pi\epsilon_0 R_2} + \cdots + \frac{Q_n}{4\pi\epsilon_0 R_n}$$

$$\therefore \boxed{V_A = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 R_m}}$$ where, $R_m = |r - r_m|$

$m = 1, 2, 3 \ldots n$

Potential gradient :-

The potential at any point due to a point charge Q is,

$$V = -\int_{\infty}^{r} \vec{E}\cdot\vec{dL} = \frac{Q}{4\pi\epsilon_0 r}$$

i.e, $V \propto \frac{1}{r}$

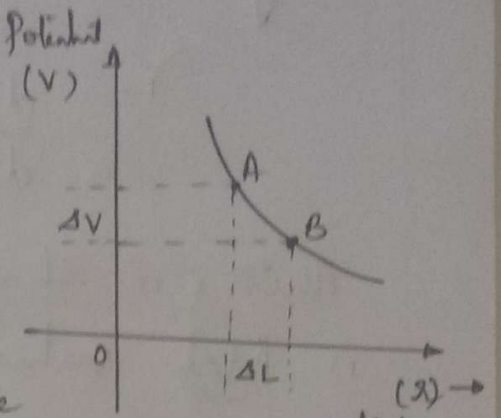


Fig :- potential gradient curve.

The potential decreases as distance from the charge increases (fig)

" The rate of change of potential with respect to distance is called the potential gradient."

i.e $\boxed{\frac{dV}{dL} = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \text{Potential gradient}}$

Relation between $\vec{E}$ & V :-

W.K.T $V = -\int \vec{E}\cdot\vec{dL}$

or $dV = -\vec{E}\cdot\vec{dL}$

$dV = -|\vec{E}||\vec{dL}|\cos\theta$

or $dV = -E\, dL \cos\theta$

$$\frac{dV}{dL} = -E\cos\theta$$

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The potential gradient is maximum i.e, $\frac{dV}{dL}$ is max

Only when $\underline{\theta = 180^\circ}$

i.e, $\left.\frac{dV}{dL}\right|_{max} = E$

This eqn shows that

1) Max value of the potential gradient gives the magnitude of the EFI.

2) Max value of potential gradient is possible only when $\vec{dL}$ is opposite to the direction of $\vec{E}$ (i.e $\theta = 180^\circ$)

$$\therefore \quad \vec{E} = -\left.\frac{dV}{dL}\right|_{max}$$

Mathematically potential gradient can be written as

gradient of V = grad $V = \nabla V$.

$$\therefore \boxed{\vec{E} = -\nabla V}$$

gradient of V is a vector quantity.

The ∇V in 3 co-ordinate systems are.

RCS – $\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$

CCS – $\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$.

SCS – $\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$.

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Current & Current density :-

"Current is defined as the rate of flow of charge carriers." It is measured in amps (A).

$$\boxed{I = -\frac{dQ}{dt}}$$

-ve sign indicates decrease in charge.

"Current density is defined as the current passing through the unit surface area, when the surface is held normal to the direction of the current."

The current density is measured in A/m2.

$$\boxed{\vec{J} = \frac{dI}{ds}\hat{a}_N}$$

where, $a_N \rightarrow$ unit vector normal to the surface $\vec{ds}$.

or $$\boxed{I = \int \vec{J} \cdot \vec{ds}}$$

Continuity equation of current :-

The continuity eqn of current is based on the principle of conservation of charge.

The total current I crossing the closed surface,

$$I = \int \vec{J} \cdot \vec{ds} \quad \text{------ (1)}$$

Also W.K.T, $$I = -\frac{dQ}{dt} \quad \text{------ (2)}$$ rate of outward flow of charge carriers

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equating ① & ②

$$\int \vec{J} \cdot \vec{ds} = -\frac{dQ}{dt} \quad \text{—— ③}$$

But $\quad Q = \int \rho_v \, dV \quad$ where $\rho_v \rightarrow$ volume charge density.

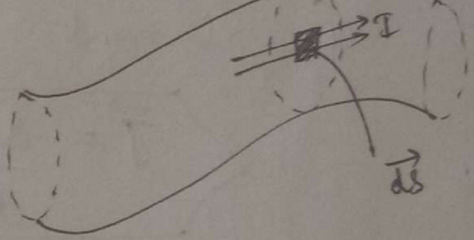
$$\int \vec{J} \cdot \vec{ds} = -\frac{d}{dt} \int \rho_v \, dv$$

$$\int \vec{J} \cdot \vec{ds} = -\int \frac{\partial \rho_v}{\partial t} \, dv \quad \text{—— ④}$$

[interchanging differential & integral operator

Using divergence theorem, converting surface integral into a volume integral,

$$\int \vec{J} \cdot \vec{ds} = \int_v (\nabla \cdot \vec{J}) \, dv$$



$\therefore$ eqn ④ becomes,

$$\int (\nabla \cdot \vec{J}) \, dv = -\int \frac{\partial \rho_v}{\partial t} \, dv$$

$$\Rightarrow \boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}} \quad \text{—— eqn of continuity.}$$

<u>Statement</u> :- "The current diverging from a small volume is equal to the time rate of decrease of charge per unit volume."

For steady current, $\nabla \cdot \vec{J} = 0$ i.e, the rate of flow of charge remains constant with time.

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Module-3 Poisson's and Laplace equations

* **Introduction:-**

We can find $\vec{E}$ & $\vec{D}$ in the given region using Coulomb's law & Gauss's law. These law's can be used only if the charge distribution is known. Practically it is not possible to know the charge distribution throughout the region. Also potential may be known only at some boundaries of the region. From this boundary values, we need to find out $\vec{E}$, $\vec{D}$ & also charge distribution. To solve this boundary value problems poisson's & Laplace equations must be known.

** **Derivation of poisson's and Laplace's equation:-**

Gauss's law in point form or Differential form [or Maxwell's first equation] allows us to derive an important differential equation for potential.

W.K.T $\nabla \cdot \vec{D} = \rho_v$ —— ①

where $\rho_v \rightarrow$ volume charge density.

Also $\vec{D} = \epsilon \vec{E}$

$\therefore$ Eqn ① becomes

$$\nabla \cdot \epsilon \vec{E} = \rho_v$$

$$\Rightarrow \nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon}$$

Using the relation $\vec{E} = -\nabla V$

$$\nabla \cdot (-\nabla V) = \frac{\rho_v}{\epsilon}$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon}$$

$$\text{or } \nabla^2 V = -\frac{\rho_v}{\epsilon} \quad \text{—— ②}$$

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The above eqn is called poisson's equation.

If $\rho_v = 0$ in certain region [charge free]

then eqn ② becomes.

$$\boxed{\nabla^2 V = 0} \quad \text{——③}$$ This is the special case

of poisson's eqn & is called Laplace's equation.

The $\nabla \cdot \nabla$ operation is called ∇^2 operation.

* Laplace eqn for 3 co-ordinate systems :-

1) Cartesian Co-ordinate System

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

2) For Cylindrical Co-ordinate System

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

3) For Spherical Co-ordinate System

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial V}{\partial \theta}\right] + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

* Procedure for solving laplace equation :-

1) Solve the Laplace equation using the method of Integration. Assume constants of Integration as per the requirement

2) Determine the constants by applying the boundary conditions given.

3) $\vec{E}$ can be obtained from the relation $\vec{E} = -\nabla V$ & also $\vec{D} = \epsilon \vec{E}$.

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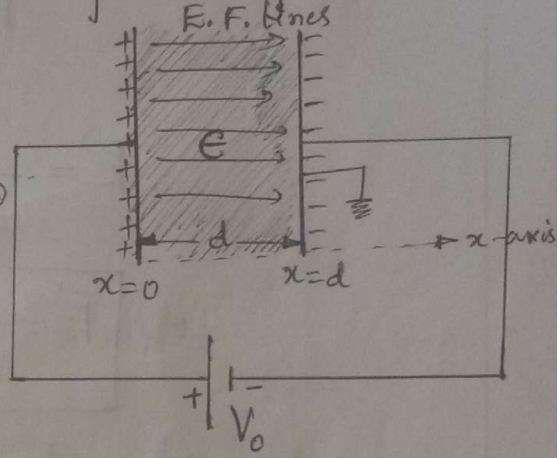
4) At the surface of the conductor, $|D_N| = \rho_s$. Hence charge on the conductor surface can be obtained as

$$\rho_s = \frac{Q}{A} \Rightarrow \boxed{Q = \rho_s A}$$

5) Once Q is known & V is known, the capacitance C of the system can be obtained.

* Applications of Laplace's equation :—

1) To find the capacitance of parallel plate capacitor.



Consider the Laplace's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{—①}$$

Since V is a function of 'x'
2nd & 3rd term is zero.

$$\Rightarrow \nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0$$

or $\nabla^2 V = \dfrac{d^2 V}{dx^2} = 0$ (only one variable)

Integrating on both side w.r.t 'x'

$\dfrac{dV}{dx} = C_1$ where C_1 is the constant of integration.

Integrating again,

$V = C_1 x + C_2$ —② where C_2 is the constant of integration.

From the boundary condition,

At $x = 0$, $V = V_0$

$x = d$, $V = 0$

$\therefore \boxed{V_0 = C_2}$ & $0 = C_1 \times d + C_2$

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$C_1 \times d = -C_2$

$C_1 d = -V_0$ or $\boxed{C_1 = -\frac{V_0}{d}}$

Substituting C_1 & C_2 in eqn ②,

$$V = -\frac{V_0}{d} x + V_0 \quad \text{③}$$

Using the relation, $\vec{E} = -\nabla V$

$$\vec{E} = -\frac{\partial V}{\partial x} \hat{a}_x$$ (∵ V is only the function of x)

$$\vec{E} = -\left[-\frac{V_0}{d}\right] \hat{a}_x$$

or $$\boxed{\vec{E} = \frac{V_0}{d} \hat{a}_x}$$

$$\therefore \vec{D} = \epsilon \vec{E} = \frac{\epsilon V_0}{d} \hat{a}_x$$

At the surface $|D_N| = \rho_s$

$$\therefore \frac{\epsilon V_0}{d} = \rho_s$$

The charge on the conductor surface,

$Q = \rho_s A$ [∵ $\rho_s = \frac{Q}{A}$ Q → charge on the plates, A → Area of the plate]

or $$Q = \frac{\epsilon V_0}{d} A$$

or $$\frac{Q}{V_0} = \frac{\epsilon A}{d}$$ But $C = \frac{Q}{V_0}$ capce of a capr

$$\therefore C = \frac{\epsilon A}{d} \Rightarrow \boxed{C = \frac{\epsilon_0 \epsilon_r A}{d}}$$ Farad.

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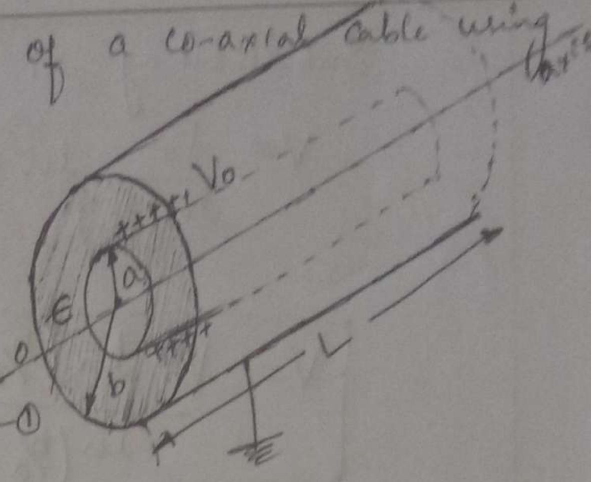
2) Derive the expression for Capacitance of a co-axial cable using the Laplace eqn.



Consider the Laplace's Equation,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{—(1)}$$

Since V is a function of 'r'

2nd & 3rd term is zero.

$$\therefore \nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] = 0$$

$$\text{or } \nabla^2 V = \frac{1}{r}\frac{d}{dr}\left[r\frac{dV}{dr}\right] = 0$$

$$\Rightarrow \frac{d}{dr}\left[r\frac{dV}{dr}\right] = 0$$

Integrating twice we get.

$$r\frac{dV}{dr} = C_1 \quad \text{or} \quad \frac{dV}{dr} = \frac{C_1}{r}$$

$$\text{or } \boxed{V = C_1 \ln r + C_2} \quad \text{—(2)}$$

Where C_1 & C_2 are constants of integration.

Using boundary conditions,

At $r = a$ $V = V_0$

$r = b$ $V = 0$

$$V_0 = C_1 \ln a + C_2 \quad \& \quad 0 = C_1 \ln b + C_2$$

$$\text{or } C_2 = -C_1 \ln b$$

$$\Rightarrow V_0 = C_1 \ln a - C_1 \ln b$$

$$V_0 = C_1 \ln(a/b)$$

$$\text{or } \boxed{C_1 = \frac{V_0}{\ln(a/b)}}$$

$$\therefore \boxed{C_2 = -\frac{V_0}{\ln(a/b)}\ln(b)}$$

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Substituting C_1 & C_2 in eqn (2),

$$V = \frac{V_0}{\ln(a/b)} \ln r - \frac{V_0}{\ln(a/b)} \ln b$$

$$\boxed{V = \frac{V_0 \ln(b/r)}{\ln(b/a)}} \quad \text{—— (3)}$$

Using the relation $\vec{E} = -\nabla V$

$$\vec{E} = -\frac{\partial V}{\partial r} \hat{a}_r$$

$$= -\frac{V_0}{\ln(b/a)} \times -\frac{1}{r} \hat{a}_r$$

$$\therefore \vec{E} = \frac{V_0}{r \ln(b/a)} \hat{a}_r$$

W.K.T $\vec{D} = \epsilon \vec{E} = \frac{\epsilon V_0}{r \ln(b/a)} \hat{a}_r$

but at the surface $|D_N| = \rho_s$

$$\therefore \rho_s = \frac{\epsilon V_0}{r \ln(b/a)}$$

The charge on the conductor surface,

$$Q = \rho_s A \qquad \because \rho_s = \frac{Q}{A}$$

$$Q = \frac{\epsilon V_0}{r \ln\left(\frac{b}{a}\right)} \times A$$

where A is a area of the cylindrical surface $A = 2\pi r L$

$$Q = \frac{\epsilon V_0}{\cancel{r} \ln\left(\frac{b}{a}\right)} \times 2\pi \cancel{r} L$$

or $\frac{Q}{V_0} = \frac{2\pi\epsilon L}{\ln(b/a)} \Rightarrow \boxed{C = \frac{Q}{V_0} = \frac{2\pi\epsilon L}{\ln(b/a)}}$ Farad

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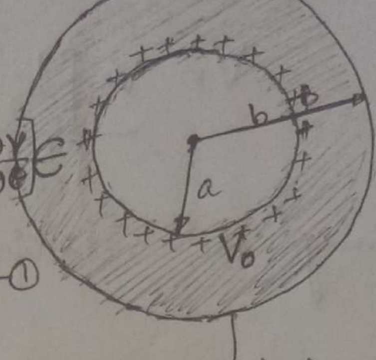
3) Find the capacitance b/w two concentric spheres of radii $r=b$ & $r=a$ such that $b>a$, if the potential $V=0$ at $r=b$, $V=V_0$ at $r=a$ using Laplace equation.

Consider the Laplace's Eqns.



$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \quad \text{--- (1)}$$

Since 'V' is a function of 'r' 2nd & 3rd terms is zero

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = 0$$

$$\text{or } \nabla^2 V = \frac{d}{dr}\left[r^2\frac{dV}{dr}\right] = 0$$

Integrating on b.s w.r.t r

$$r^2\frac{dV}{dr} = C_1 \implies \frac{dV}{dr} = \frac{C_1}{r^2}$$

Integrating again,

$$\boxed{V = -\frac{C_1}{r} + C_2} \quad \text{--- (2)}$$

Where C_1 & C_2 are constants of integration.

Using boundary conditions,

At $r=a$, $V=V_0$

$r=b$ $V=0$

$$0 = \frac{-C_1}{b} + C_2 \quad \& \quad V_0 = \frac{-C_1}{a} + C_2$$

$$C_2 = \frac{C_1}{b}$$

$$V_0 = -\frac{C_1}{a} + \frac{C_1}{b} = C_1\left[\frac{1}{b} - \frac{1}{a}\right]$$

$$\boxed{C_2 = \frac{V_0}{b\left(\frac{1}{b}-\frac{1}{a}\right)}} \qquad \boxed{C_1 = \frac{V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)}}$$

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Substituting C_1 & C_2 in eqn (2).

$$V = \frac{-V_0}{r\left(\frac{1}{b}-\frac{1}{a}\right)} + \frac{V_0}{b\left(\frac{1}{b}-\frac{1}{a}\right)}$$

$$V = \frac{V_0\left[\frac{1}{r}-\frac{1}{b}\right]}{\left[\frac{1}{a}-\frac{1}{b}\right]} \quad \text{—— (3)}$$

Using the relation $\vec{E} = -\nabla V$

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{a}_r \quad (\because V \text{ is a fun of } r \text{ only})$$

$$\vec{E} = -\frac{V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)} \times -\frac{1}{r^2}\hat{a}_r$$

$$\vec{E} = \frac{V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}\hat{a}_r$$

W.K.T $\vec{D} = \epsilon\vec{E} = \frac{\epsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}\hat{a}_r$

But @ the surface $|D_N| = \rho_s$ $\quad \therefore \rho_s = \frac{\epsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}$

The charge on the conductor surface.

$$Q = \rho_s A \qquad \left[\because \rho_s = \frac{Q}{A}\right]$$

$$Q = \frac{\epsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} A$$

where $A = 4\pi r^2$ is the area of the spherical surface.

$$Q = \frac{\epsilon V_0}{\cancel{r^2}\left(\frac{1}{a}-\frac{1}{b}\right)} \times 4\pi \cancel{r^2}$$

$$\frac{Q}{V_0} = \frac{4\pi\epsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)} \Rightarrow \boxed{C = \frac{Q}{V_0} = \frac{4\pi\epsilon ab}{(b-a)}} \text{ farad}$$

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* Uniqueness theorem :-

" The Uniqueness theorem can be stated as follows, If a solution to Laplace's eqn or poisson's eqn can be found out that satisfies the boundary conditions then the solution is unique."

Let us assume two possible solutions V_1 & V_2 (or) Laplace eqn has two solutions say V_1 & V_2.

i.e, $\nabla^2 V_1 = 0$ & $\nabla^2 V_2 = 0$

Both the solutions must satisfy the boundary conditions as well.

$$\nabla^2 (V_1 - V_2) = 0 \quad \text{——} ①$$

$$\text{or} \quad V_1 - V_2 = 0 \quad \text{——} ②$$

i.e, $V_1 = V_2$ along the boundary surface.

Proof:- The divergence theorem of any vector $\vec{F}$

$$\oint_S \vec{F} \cdot \vec{ds} = \int_V (\nabla \cdot \vec{F})\, dV$$

Let $\vec{F}$ be $\vec{F} = \underbrace{(V_1 - V_2)}_{\alpha} \underbrace{\nabla (V_1 - V_2)}_{B}$

Using vector identity,

$$\nabla \cdot (\underbrace{\alpha \vec{B}}_{\vec{F}}) = \alpha (\nabla \cdot \vec{B}) + \vec{B} \cdot (\nabla \alpha)$$

$$\therefore \oint_S (V_1 - V_2) \nabla (V_1 - V_2) \cdot \vec{ds} = \int_V (V_1 - V_2)\, \nabla \cdot \nabla (V_1 - V_2) + \nabla (V_1 - V_2) \cdot \nabla (V_1 - V_2) \cdot dV$$

$$\cancel{(LHS)}^{0} = \int_V \cancel{(V_1 - V_2) \nabla^2 (V_1 - V_2)}^{0}\, dV + \int_V [\nabla (V_1 - V_2)]^2\, dV$$

Using ① & ② LHS & RHS 1st term is zero

$$\therefore \int_V [\nabla(V_1 - V_2)]^2 dV = 0$$

The square of the gradient is zero, if and only if the gradient it self is zero.

$$\nabla(V_1 - V_2) = 0$$

$$\Rightarrow V_1 - V_2 = \text{constant}$$

Thus the constant must be zero or $V_1 = V_2$ everywhere

Thus in a given region, Laplace eqn has only one solution that satisfies the boundary conditions.

Problems :-

1) Determine whether or not the following potential fields satisfies the Laplace eqn.

a) $V = x^2 + y^2 + z^2$ b) $V = r\cos\phi + z$ c) $V = r\cos\theta + \phi$

d) $V = 2x^2 - 3y^2 + z^2$ e) $V = x^2 - y^2 + z^2$ f) $V = r^2\cos\phi + \theta$

g) $V = r\cos\phi + z$ h) $V = 2x^2 - 4y^2 + z^2$.

2) Given that $V = XY$ is a solution of Laplace's eqn, where X is a fun of x alone & Y is a fun of y alone. Determine which of the folly potential funs are also solns of Laplace's Eqn.

→ i) $V = 100X$ ii) $V = 80XY$ iii) $V = 3XY + x - by$.

→ i) $V = 100X$ ii) $V = 50XY$ iii) $V = 2XY - x - 3y$.

3) Given $V = A \ln\left[B\dfrac{(1-\cos\theta)}{1+\cos\theta}\right]$

i) S.T 'V' satisfies Laplace Eqn in spherical Co-ordinates.

ii) Find A & B so that $V = 100\,V$, $|E| = 500\,V/m$ @ $r = 5m$, $\theta = 90°$ & $\phi = 60°$

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Module-3 Steady Magnetic field

The study of steady magnetic field produced due to the flow of direct current through a conductor is called Magneto statics.

According to Oersted, a current carrying conductor is surrounded by a magnetic field. It can be represented by magnetic lines of force. These lines of force are also called magnetic flux lines. It always exist in the form of concentric circles (closed loop).

* Magnetic field intensity :- [$\vec{H}$]

"Magnetic field intensity at any point in the magnetic field is defined as the force experienced by 1wb when placed at that point."

It is measured in N/wb or A/m.

* Magnetic flux Density [$\vec{B}$] :-

"The magnetic flux crossing a unit surface area, when the area is normal to the flux lines. It is denoted as $\vec{B}$ and is measured in wb/m^2 or Tesla (T)."

$$\vec{B} = \frac{d\phi}{ds} \hat{a}_N$$

$d\phi$ is the magnetic flux lines passing through

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the area ds.

$$\boxed{\phi = \int_{open} \vec{B} \cdot \vec{ds}} \text{ wb}$$

* **Relation between $\vec{B}$ & $\vec{H}$:-**

The $\vec{B}$ & $\vec{H}$ are related as,

$$\vec{B} = \mu \vec{H}$$

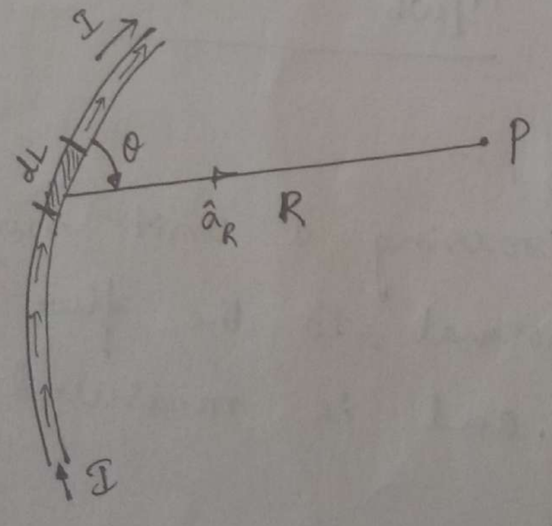
where $\mu = \mu_0 \mu_r$

$\mu_0 = 4\pi \times 10^{-7}$ H/m is the permeability of free space.

$$\boxed{\vec{B} = \mu_0 \mu_r \vec{H}}$$

Imp * **Biot - Savart law :-**

Consider a conductor carrying a direct current I & a steady magnetic field produced around it.



Consider a differential length dL which is very small part of the current-carrying conductor. The point P is at a distance R from the differential current element.

Biot - Savart law states that.

" The magnetic field intensity $\vec{dH}$ produced at a point P due to dL is

1) Proportional to the product of current I & differential length dL.

2) The sine of the angle b/w the element & the line joining point P.

3) And inversely proportional to the square of the distance R b/w point P & the element.

i.e., $$\vec{dH} \propto \frac{I\,dL\,\sin\theta}{R^2}$$

or $$\vec{dH} = \frac{I\,dL\,\sin\theta}{4\pi R^2}$$

From the cross product rule,

$$\vec{dL} \times \hat{a}_R = |dL|\,|\hat{a}_R|\,\sin\theta = dL\,\sin\theta.$$

$$\therefore \boxed{\vec{dH} = \frac{I\,\vec{dL} \times \hat{a}_R}{4\pi R^2}}$$

But $\hat{a}_R = \dfrac{\vec{R}}{|\vec{R}|}$

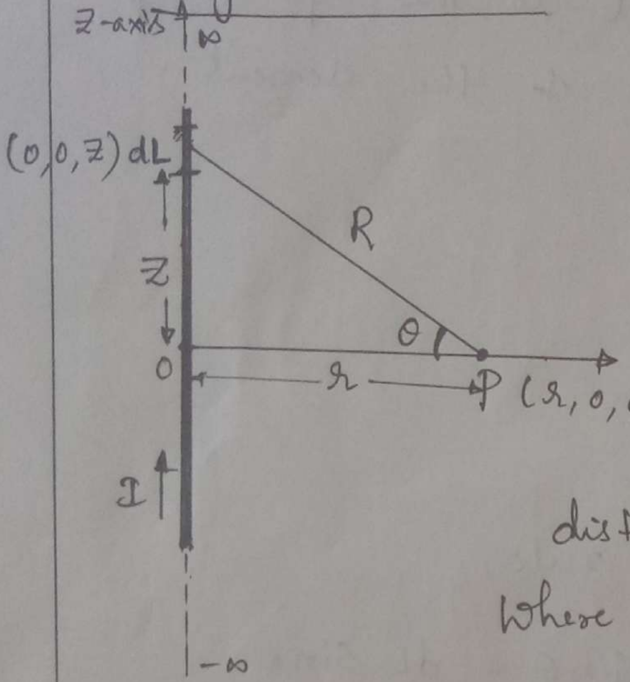
The total magnetic field intensity $\vec{H}$ due to all such dL's are, (differential elements) is given by,

$$\boxed{\vec{H} = \oint \frac{I\,\vec{dL} \times \hat{a}_R}{4\pi R^2}} \quad A/m$$

But $I\,\vec{dL} = \vec{J}\,dV$

$$\therefore \boxed{\vec{H} = \int \frac{\vec{J} \times \hat{a}_R}{4\pi R^2}\, dv} \quad A/m$$

** Magnetic field intensity ($\vec{H}$) due to infinitely long Straight conductor :-



Consider an infinitely long straight conductor, along z-axis. The current through the conductor is I. Let P be a point at a distance 'r' from the Z-axis, where $\vec{H}$ is to be calculated.

Consider a differential element dL along z-axis, at a distance Z from the Origin.

According to Biot-Savart law,

$\vec{dH}$ at point P is given by,

$$\vec{dH} = \frac{I\, \vec{dL} \times \hat{a}_R}{4\pi R^2}.$$

where $\hat{a}_R = \frac{\vec{R}}{|\vec{R}|}$, But $\vec{R} = r\,\hat{a}_r - z\,\hat{a}_z$

$$|\vec{R}| = \sqrt{r^2 + z^2}$$

or R

$$\hat{a}_R = \frac{r\,\hat{a}_r - z\,\hat{a}_z}{\sqrt{r^2 + z^2}}$$

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$$\vec{dL} \times \hat{a}_R = \frac{1}{\sqrt{r^2+z^2}} \begin{vmatrix} \hat{a}_r & \hat{a}_\phi & \hat{a}_z \\ 0 & 0 & dz \\ r & 0 & -z \end{vmatrix}$$

$\{ \vec{dL} = dz$ … $\because$ on z-

$$= \frac{r\,dz\,\hat{a}_\phi}{\sqrt{r^2+z^2}}$$

$$\therefore \vec{dH} = \frac{I\,r\,dz\,\hat{a}_\phi}{4\pi R^2\sqrt{r^2+z^2}} = \frac{I\,r\,dz}{4\pi(r^2+z^2)^{3/2}}\hat{a}_\phi$$

Thus total field intensity $\vec{H}$ can be obtained by integrating $\vec{dH}$ over the entire length.

$$\vec{H} = \int_{z=-\infty}^{\infty} \vec{dH} = \int_{-\infty}^{\infty} \frac{I\,r\,dz}{4\pi(r^2+z^2)^{3/2}}$$

From the fig. $\tan\theta = \frac{z}{r} \Rightarrow \boxed{z = r\tan\theta}$

$$z^2 = r^2\tan^2\theta$$

$$\boxed{dz = r\sec^2\theta\,d\theta}$$

If $z = -\infty$, $\theta = -\pi/2$

$z = \infty$, $\theta = \pi/2$

$$\therefore \vec{H} = \int_{\theta=-\pi/2}^{\pi/2} \frac{I\,r \times r\sec^2\theta\,d\theta}{4\pi(r^2+r^2\tan^2\theta)^{3/2}} = \int_{-\pi/2}^{\pi/2} \frac{I\,r^2\sec^2\theta\,d\theta}{4\pi r^3\sec^3\theta}$$

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$$\vec{H} = \int_{-\pi/2}^{\pi/2} \frac{I}{4\pi r} \cos\theta \, d\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\frac{1}{\sec\theta} = \cos\theta$$

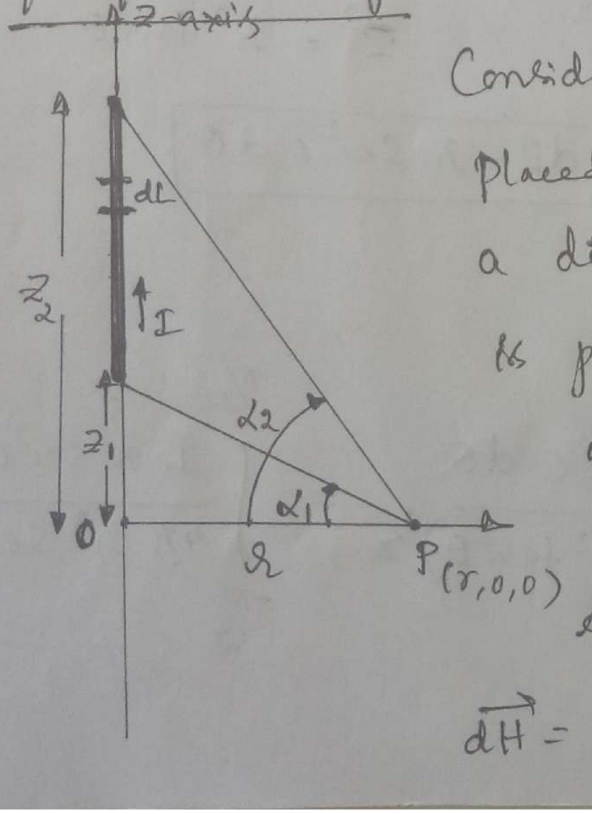
$$= \frac{I}{4\pi r} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{I}{4\pi r} \left[\sin\theta \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{I}{4\pi r} \left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) \right] = \frac{I}{4\pi r} \left[1 - (-1) \right]$$

$$\boxed{\vec{H} = \frac{I}{2\pi r} \hat{a}_\phi} \quad A/m$$

W.K.T $\boxed{\vec{B} = \mu \vec{H} = \frac{\mu I}{2\pi r} \hat{a}_\phi}$ wb/m^2

** Magnetic field intensity $\vec{H}$ due to straight conductor of finite length



Consider a conductor of finite length placed along Z-axis. It carries a direct current I. The conductor is placed such that its one end is at $Z = Z_1$ while other at $Z = Z_2$.

According to Biot-Savart law,

$$\vec{dH} = \frac{I \, \vec{dL} \times \hat{a}_R}{4\pi R^2}$$

Where $\hat{a}_R = \frac{\vec{R}}{|\vec{R}|}$ $\quad \vec{R} = r\,\hat{a}_r - z\,\hat{a}_z$

$|\vec{R}| = R = \sqrt{r^2 + z^2}$

$$\therefore \hat{a}_R = \frac{r\,\hat{a}_r - z\,\hat{a}_z}{\sqrt{r^2+z^2}}$$

then $\vec{dL} \times \hat{a}_R = \frac{r\,dz\,\hat{a}_\phi}{\sqrt{r^2+z^2}}$

$$\therefore \vec{dH} = \frac{I\,r\,dz\,\hat{a}_\phi}{4\pi(r^2+z^2)\sqrt{r^2+z^2}} = \frac{I\,r\,dz}{4\pi\,(r^2+z^2)^{3/2}}\,\hat{a}_\phi$$

The total $\vec{H}$ at P due to conductor of finite length,

$$\vec{H} = \int_{z_1}^{z_2} \vec{dH} = \int_{z_1}^{z_2} \frac{I\,r\,dz}{4\pi\,(r^2+z^2)^{3/2}}$$

From the fig. $\tan\alpha_1 = \frac{z_1}{r} \Rightarrow z_1 = r\tan\alpha_1$

$z_2 = r\tan\alpha_2$

$$\therefore Z = r\tan\alpha$$

$$dz = r.\sec^2\alpha\,d\alpha.$$

$$\vec{H} = \int_{\alpha_1}^{\alpha_2} \frac{I\,r\,r\sec^2\alpha\,d\alpha}{4\pi\,(r^2+r^2\tan^2\alpha)^{3/2}} = \int_{\alpha_1}^{\alpha_2} \frac{I}{4\pi r}\cos\alpha\,d\alpha$$

$$= \frac{I}{4\pi r}\int_{\alpha_1}^{\alpha_2} \cos\alpha\,d\alpha$$

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$$\vec{H} = \frac{I}{4\pi r}\left[\sin\alpha\right]_{\alpha_1}^{\alpha_2}$$

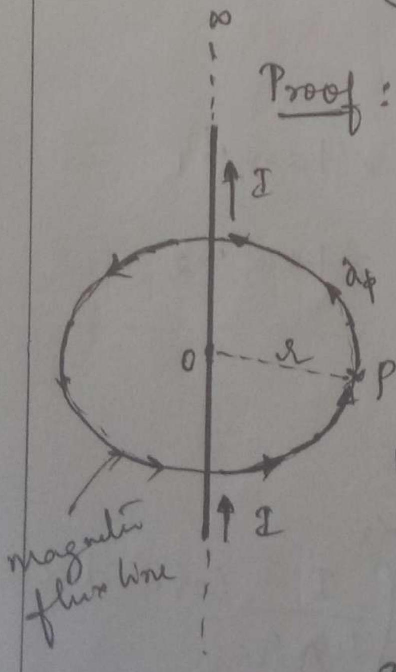
$$\boxed{\vec{H} = \frac{I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right]\hat{a}_\phi} \quad A/m.$$

$$\boxed{\vec{B} = \mu\vec{H} = \frac{\mu I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right]\hat{a}_\phi} \quad wb/m^2$$

** Ampere's Circuital law :-

"Ampere's Circuital law states that the line integral of $\vec{H}$ about any closed path is exactly equal to the direct current enclosed by that path."

i.e., $$\boxed{\oint \vec{H}\cdot\vec{dL} = I}$$



Proof :-

Consider a closed circular path of radius r which encloses the straight conductor carrying the current I.

W.K.T $\vec{H} = H_\phi\,\hat{a}_\phi$ ($\vec{H}$ has only component in $\hat{a}_\phi$ direction)

$\vec{dL} = r\,d\phi\,\hat{a}_\phi$ (in the ϕ direction)

Substituting we get

$$\oint H_\phi\,\hat{a}_\phi \cdot r\,d\phi\,\hat{a}_\phi = I$$

$$r H_\phi \int_0^{2\pi} d\phi = I \qquad (\because \hat{a}_\phi \cdot \hat{a}_\phi = 1)$$

or $\quad r H_\phi [\phi]_0^{2\pi} = I \Rightarrow r H_\phi \, 2\pi = I$

$$\therefore \boxed{H_\phi = \frac{I}{2\pi r}}$$

Hence $\vec{H}$ at any pt P is given by,

$$\boxed{\vec{H} = \frac{I}{2\pi r} \hat{a}_\phi} \quad A/m$$

* Physical Significance of Curl :-

The curl of vector at any point indicates the circulation of vector at that point. In general

If there is no rotation, no curl.

Curl always exists for rotational fields.

$\therefore$ Curl of $\vec{H}$ is given by,

$$\boxed{\nabla \times \vec{H} = \lim_{\Delta S \to 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta S}}$$

"Curl of $\vec{H}$ is defined as the circulation of $\vec{H}$ per unit area as area shrinks to zero."

Curl of $\vec{H}$ in various co-ordinate systems :-

1) RCS (x, y, z)

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

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2) CCS (r, ϕ, z)

$$\nabla \times \vec{H} = \frac{1}{r} \begin{vmatrix} \overset{+}{\hat{a}_r} & \overset{-}{r\hat{a}_\phi} & \overset{+}{\hat{a}_z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

3) SCS (r, θ, ϕ)

$$\nabla \times \vec{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \overset{+}{\hat{a}_r} & \overset{-}{r\hat{a}_\theta} & \overset{+}{r\sin\theta\, \hat{a}_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta\, H_\phi \end{vmatrix}$$

Properties of curl :-

1) The divergence of a curl is zero.

i.e., $\nabla \cdot (\nabla \times \vec{A}) = 0$

2) The curl of gradient of V is zero.

i.e., $\nabla \times \nabla V = 0$.

3) $\nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Stokes theorem :-

Consider the Ampere's Circuital law,

$$\oint \vec{H} \cdot d\vec{L} = I \quad \text{——} (1)$$

$\div \Delta S$ on both side

$$\frac{\oint \vec{H} \cdot d\vec{L}}{\Delta S} = \frac{I}{\Delta S}$$

where ΔS is the area enclosed by the perimeter (closed path) as ΔS is small, taking lim as $\Delta S \to 0$ on both side,

$$\lim_{\Delta S \to 0} \frac{\oint \vec{H} \cdot \vec{dL}}{\Delta S} = \lim_{\Delta S \to 0} \frac{I}{\Delta S}$$

But $\lim_{\Delta S \to 0} \frac{I}{\Delta S} = \vec{J}$, current density

$$\therefore \boxed{\nabla \times \vec{H} = \vec{J}} \quad \text{--- (2)}$$

Ampere's circuital law in point form or differential form.

W.K.T Current density $\vec{J} = \frac{dI}{dS} \hat{a}_N$

or $I = \int \vec{J} \cdot \vec{ds}$

using eqn (2)

$$I = \int (\nabla \times \vec{H}) \cdot \vec{ds} \quad \text{--- (3)}$$

from (1) & (3), we can write

$$\boxed{\oint \vec{H} \cdot \vec{dL} = \int_{open} (\nabla \times \vec{H}) \cdot \vec{ds}}$$

" The line integral of $\vec{H}$ around any closed path is equal to the surface integral of curl of $\vec{H}$ enclosed by the same closed path."

Using Stoke's theorem, line integral can be

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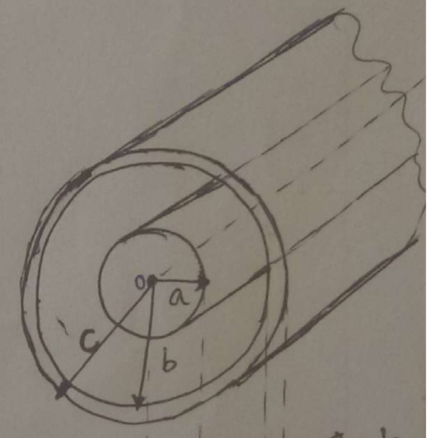
Converted into Surface Integral.

* Note :- Stoke's theorem is applicable for the open surface enclosed by the given closed path.

* $\vec{H}$ due to a co-axial cable :-

Consider a co-axial cable, its inner conductor carrying current I & of radius 'a'. The outer conductor is in the form of concentric cylinder whose inner radius is b & outer radius c.

Region The space b/w inner and outer conductor is filled with dielectric.



Region 1 :- Within the inner conductor $(r<a)$. Consider a closed path of radius $r<a$. Hence it covers only part of the conductor.

Therefore the current enclosed by the closed path is,

$$I' = \frac{r^2}{a^2} I$$

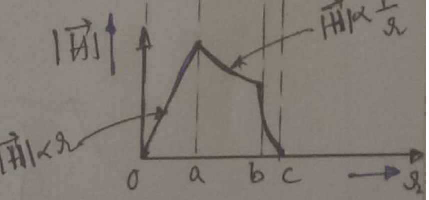


fig:- Variation of $\vec{H}$ against 'r' in co-axial cable

from ampere's Circuital law,

$$\vec{H} = \frac{I'}{2\pi r} \hat{a}_\phi = \frac{r^2}{2\pi r a^2} I = \frac{I r}{2\pi a^2} \hat{a}_\phi$$

$$\vec{H} = \frac{I r}{2\pi a^2} \hat{a}_\phi \quad \text{—①}$$

Region 2 :- Within $a < r < b$, the closed path encloses the Inner Conductor carrying direct current I.

$$\therefore \quad \boxed{\vec{H} = \frac{I}{2\pi r}\,\hat{a}_\phi} \quad \text{—— (2)}$$

Region 3 :- Within Outer Conductor, $b < r < c$, the current enclosed by the closed by is

$$I' = I\frac{[c^2 - r^2]}{[c^2 - b^2]}$$

From, ampere's Circuital law,

$$\vec{H} = \frac{I'}{2\pi r}\,\hat{a}_\phi$$

$$\boxed{\vec{H} = \frac{I}{2\pi r}\frac{[c^2 - r^2]}{[c^2 - b^2]}\,\hat{a}_\phi} \quad \text{—— (3)}$$

Region 4 :- Outside the cable , $r > c$

$$\boxed{\vec{H}_\phi = 0}$$

The magnetic field does not exist outside the cable.

—— x ——

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Module-4 Magnetic Forces

* Force on a Moving point Charge :-

Consider a Static electric field $\vec{E}$, the force that is exerted on a static or moving charge Q is given by

$$\boxed{\vec{F_e} = Q\vec{E}} \text{ Newtons}$$

The direction of the force is always in the direction of $\vec{E}$. This force is referred as Electric force ($\vec{F_e}$).

Consider a steady magnetic field, it is capable of producing a force only on a dynamic (moving) charge,

$$\boxed{\vec{F_m} = Q\vec{v} \times \vec{B}} \text{ Newtons.}$$

This magnetic force is always ⊥r to the magnetic field.

The magnitude of the force is

$$|\vec{F_m}| = F_m = QvB\sin\theta$$

The magnetic force depends on Q, v & B and also Sine of the angle between $\vec{v}$ and $\vec{B}$.

The net (total) force on a moving charge Q when both electric & magnetic field exists.

$$\vec{F} = \vec{F_e} + \vec{F_m}$$

$$\vec{F} = Q\vec{E} + Q\vec{v} \times \vec{B}$$

$$\boxed{\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})} \text{ Newtons}$$

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This equation is called Lorentz force equation. which relates mechanical force to electrical force.

Force on a differential current element :-

The differential magnetic force on a differential charge dQ moving with a velocity $\vec{v}$ is given by,

$$d\vec{F}_m = dQ(\vec{v} \times \vec{B}) \quad \text{—— (1)}$$

The current in a conductor can be expressed as charge per unit time

$$I = \frac{Q}{t}$$

xly by 'l' on both side,

$$Il = Q\left(\frac{l}{t}\right)$$

Where $\vec{v} = \frac{l}{t}$ is the velocity of the charge Q to move through a distance l.

$$\Rightarrow Il = Q\vec{v}$$

$\therefore$ for a differential length $d\vec{L}$,

$$\boxed{dLI = dQ\,\vec{v}}$$

Using this eqn (1) becomes,

$$d\vec{F}_m = dLI \times \vec{B}$$

or $$\boxed{d\vec{F}_m = I\,d\vec{L} \times \vec{B}}$$

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This ~~this~~ is the force exerted on a differential current element.

From the relationship,

$$I\,d\vec{L} = \vec{J}\,dv$$

∴ The magnetic force in terms of current density $\vec{J}$ is

$$d\vec{F_m} = \vec{J}\,dv \times \vec{B}$$

or $$\boxed{d\vec{F_m} = \vec{J} \times \vec{B}\,dv}$$

If a conductor is straight, carrying current I in a uniform magnetic field $\vec{B}$, then the magnetic force on the conductor is given by,

$$\vec{F_m} = I\,\vec{L} \times \vec{B}$$

The magnitude of the force is given by.

$$F_m = ILB\sin\theta$$

* Force between differential current elements :-

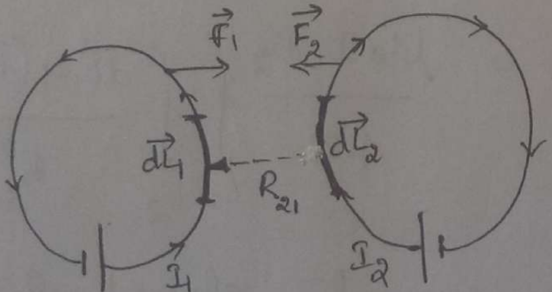


Fig: Force between two current elements

In both the loops, currents are flowing in the same direction. Consider two current elements $I_1 \vec{dL_1}$ & $I_2 \vec{dL_2}$ as shown in figure.

The force exerted on the first element due to the magnetic field produced by other element is given by,

$$d\vec{F_1} = I_1 \vec{dL_1} \times d\vec{B_2} \quad \text{——} \; ①$$

where

$$d\vec{B_2} = \mu \, d\vec{H_2}$$

According to Biot - Savart law.

$$d\vec{H_2} = \frac{I_2 \, \vec{dL_2} \times \hat{a}_{R_{21}}}{4\pi R_{21}^2}$$

∴ Eqn ① becomes,

$$d\vec{F_1} = \frac{I_1 \vec{dL_1} \times \mu \, I_2 \vec{dL_2} \times \hat{a}_{R_{21}}}{4\pi R_{21}^2}$$

The total force $\vec{F_1}$ on the current element - 1 due to current element - 2 is given by,

$$\vec{F_1} = \int d\vec{F_1} = \frac{\mu I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{\vec{dL_1} \times (\vec{dL_2} \times \hat{a}_{R_{21}})}{R_{21}^2} \quad \text{——} \; ②$$

Ily force on the current element 2 due to magnetic field produced by current element 1 is

$$\vec{F_2} = \frac{\mu I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{\vec{dL_2} \times (\vec{dL_1} \times \hat{a}_{R_{12}})}{R_{12}^2} \quad \text{—— (3)}$$

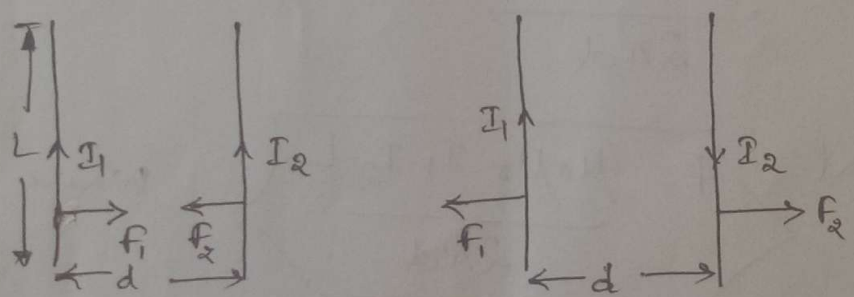
from (2) & (3)

$$\boxed{\vec{F_1} = \vec{F_2}}$$

indicates force is attractive as the current flows in the same direction.

If current flows in the opposite direction, then $\vec{F_1} = -\vec{F_2}$, i.e, the force is repulsive.

Force b/n two parallel conductors :-



Force on conductor 1 due to magnetic field produced by conductor 2 is given by,

$$\vec{F_1} = I_1 \vec{L_1} \times \vec{B_2} \quad \text{—— (1)}$$

the mag of the force is,

$$F_1 = I_1 L_1 B_2 \sin\theta \qquad \text{If } \theta = 90^\circ$$

$$F_1 = I_1 L_1 B_2 \quad \text{—— (2)}$$

But $B_2 = \mu H_2$ where $H_2 = \dfrac{I_2}{2\pi d}$

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$$\therefore \quad B_2 = \frac{\mu_0 \mu_r I_2}{2\pi d}$$

$\therefore$ Eqn ② becomes $f_1 = \frac{I_1 L \mu_0 \mu_r I_2}{2\pi d}$ or $\boxed{f_1 = \frac{\mu_0 \mu_r I_1 I_2 L}{2\pi d}}$

Illy the magnitude of the force acting on conductor 2 due to magnetic field produced by conductor 1 is given by

$$F_2 = \frac{\mu_0 \mu_r I_1 I_2 L}{2\pi d}$$

In general $\boxed{F = \frac{\mu_0 \mu_r I_1 I_2 L}{2\pi d}}$ where I_1 & I_2 are the current flowing through conductor 1 & 2 and d is the distance of separation b/n two conductors.

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Time-Varying fields and Maxwell's equations

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Module-5

Faraday's Law :-

Statement :- " The electromotive force (emf) induced in a closed path (circuit) is proportional to rate of change of magnetic flux enclosed by that closed path."

i.e, $e = -N \frac{d\phi}{dt}$ —— ①

where $N \rightarrow$ no of turns in the circuit.

$e \rightarrow$ induced emf.

Let $N=1$, then $e = -\frac{d\phi}{dt}$ volts. —— ②

The minus sign in the above eqn indicates that the induced emf opposes the cause producing it.

The induced emf about a closed path,

$$e = \int \vec{E} \cdot d\vec{L} \quad \text{—— ③}$$

The magnetic flux passing through a specified area is given by.

$$\phi = \int \vec{B} \cdot d\vec{S} \quad \text{—— ④}$$

from ③ & ④ eqn ② becomes.

$$\int \vec{E} \cdot d\vec{L} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{S}$$

$$\boxed{\int \vec{E} \cdot d\vec{L} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}} \quad \text{—— ⑤}$$

Maxwell's eqn in integral form.

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This is similar to transformer action & emf is called transformer emf.

Using Stoke's theorem, line integral can be converted into surface integral,

i.e., $\int \vec{E} \cdot \vec{dL} = \int (\nabla \times \vec{E}) \cdot \vec{dS}$

$\therefore$ eqn (5) becomes,

$$\int (\nabla \times \vec{E}) \cdot \vec{dS} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \vec{dS}$$

$$\Rightarrow \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ Maxwell's Eqn in point form or differential form.

Special case:-

If magnetic field is stationary, while the closed path is moving:-

When a closed path is moving in a stationary (static) magnetic field $\vec{B}$, this action is similar to the generator action, hence the induced emf is called motional or generator emf. (Dynamically induced emf)

The force on a charge Q moving with a velocity v in the magnetic field $\vec{B}$ is

$$\vec{F} = Q(\vec{v} \times \vec{B}) \quad \text{But } \vec{E} = \frac{\vec{F}}{Q}$$

$$\therefore \vec{E} = \vec{v} \times \vec{B}$$

Thus induced emf is given by,

$$\boxed{e = \int \vec{E} \cdot \vec{dL} = \int (\vec{v} \times \vec{B}) \cdot \vec{dL}}$$

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* **Concept of Displacement current density and displacement current**

According to Ampere's circuital law,

$$\nabla \times \vec{H} = \vec{J} \quad \text{——— ①}$$

Taking divergence on both side,

$$\underbrace{\nabla \cdot (\nabla \times \vec{H})} = \nabla \cdot \vec{J}$$ (∵ the divergence of a curl of any vector, is zero)

$$0 = \nabla \cdot \vec{J}$$

But from the continuity equation of current

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

Therefore, equation ① must be modified by adding unknown term say $\vec{G}$

$$\therefore \quad \nabla \times \vec{H} = \vec{J} + \vec{G} \quad \text{——— ②}$$

Again taking divergence on both side,

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot (\vec{J} + \vec{G})$$

$$0 = \nabla \cdot \vec{J} + \nabla \cdot \vec{G}$$

or $\nabla \cdot \vec{G} = -\nabla \cdot \vec{J}$

But from continuity eqn of current,

$$\nabla \cdot \vec{G} = -\left[-\frac{\partial \rho_v}{\partial t}\right] = \frac{\partial \rho_v}{\partial t}$$

But from Maxwell's 1st eqn

$$\nabla \cdot \vec{D} = \rho_v$$

$$\nabla \cdot \vec{G} = \frac{\partial (\nabla \cdot \vec{D})}{\partial t}$$

$$\nabla \cdot \vec{G} = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \boxed{\vec{G} = \frac{\partial \vec{D}}{\partial t}}$$

$\therefore$ Eqn (2) becomes.

$$\boxed{\nabla \times \vec{H} = J + \frac{\partial \vec{D}}{\partial t}}$$ Ampere's law for time varying field.

The 1^{st} term in above eqn is conduction current density denoted by J_c. The 2nd term in eqn represents displacement current density denoted by J_D.

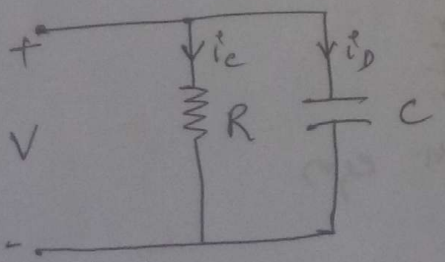
$$\therefore \boxed{\nabla \times \vec{H} = \vec{J_c} + \vec{J_D}}$$

Conduction current density $\vec{J_c} = \sigma \vec{E}$

& Displacement current density $\vec{J_D} = \frac{\partial \vec{D}}{\partial t}$.

Displacement current is associated with time varying fields & exists in all imperfect conductors.

* Physical Significance of displacement current :-



Consider a $\parallel^{el}$ circuit of resistor & capacitor driven by a time varying voltage V.

Let the current through the resistor R be i_c & the current through the capacitor C be i_D. The nature of the current flowing through the resistor is different than that flowing through the capacitor.

The current through the resistor is due to motion of charges,

$$i_c = \frac{V}{R} \quad \text{——} \; (1)$$

Let 'A' be the cross sectional area of conductor

then $$\vec{J}_c = \frac{i_c}{A} = \sigma \vec{E} \quad \text{——} \; (2)$$

The current through the capacitor is due to the displacement of charge carriers.

$$i_D = C\frac{dV}{dt} \quad \text{——} \; (3) \quad \text{where} \quad C = \frac{\epsilon A}{d}$$

$$\therefore \quad i_D = \frac{\epsilon A}{d}\frac{dV}{dt} \qquad \text{But } E = \frac{V}{d} \Rightarrow V = Ed$$

$$\therefore \quad i_D = \frac{\epsilon A}{d}\frac{d}{dt}(Ed) = \epsilon A\frac{d\vec{E}}{dt}$$

$$\Rightarrow \quad \frac{i_D}{A} = \frac{d\,\epsilon\vec{E}}{dt} \qquad \text{but } \vec{D} = \epsilon\vec{E}$$

$$J_D = \frac{\partial \vec{D}}{\partial t} \quad \text{——} \; (4)$$

$\therefore$ Total current density, $\boxed{\vec{J} = \vec{J}_c + \vec{J}_D}$.

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* ## Importance of $\vec{J_C}$ & $\vec{J_D}$:-

WKT some materials are good conductors & some are good (perfect) dielectrics. But there are some materials which are neither good conductor nor perfect dielectrics, through which both the current namely conduction current & displacement current may flow.

i.e, total current density $\vec{J} = \vec{J_c} + \vec{J_D}$

where, $\vec{J_c} = \sigma \vec{E}$ ——— ①

$$\vec{J_D} = \frac{\partial \vec{D}}{\partial t} = \frac{\partial \epsilon \vec{E}}{\partial t} = j\omega \epsilon \vec{E} \quad \text{——— ②}$$

time dependence $\frac{\partial}{\partial t}$ can be written as $j\omega$.

from ① & ②

$$\vec{J} = \sigma \vec{E} + j\omega \epsilon \vec{E}$$

And, $|\vec{J_c}| = \sigma \vec{E}$ & $|\vec{J_D}| = \omega \epsilon \vec{E}$

$$\therefore \frac{|J_c|}{|J_D|} = \frac{\sigma \vec{E}}{\omega \epsilon E} = \frac{\sigma}{\omega \epsilon}$$

$$\Rightarrow \boxed{\frac{|J_c|}{|J_D|} = \frac{\sigma}{\omega \epsilon}}$$

The ratio of the magnitudes of the conduction current density to the displacement current density depends on the properties of the medium σ & ϵ

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& the frequency (i.e, ω).

For a conductor σ is very large, ∴ conduction current is very large compared to displacement current

i.e, $\frac{\sigma}{\omega\epsilon} >> 1$ medium is conductor.

For a dielectric, the displacement current is greater compared to the conduction current.

i.e, $\frac{\sigma}{\omega\epsilon} << 1$ medium is dielectric

** **Maxwell's Equations :-**

v.imp

Maxwell's eqns are 4 std equations derived from Ampere's law, Faraday's law, Gauss's law for electric field & Gauss's law for magnetic field.

1) Faraday's law :-

According to Faraday's law, the induced emf around a closed path is given by,

$$e = -\frac{d\phi}{dt} \quad \text{but } \phi = \int_s \vec{B}\cdot\vec{ds}$$

e = V (voltage)

$$\Longrightarrow e = -\int \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds}$$

$$\boxed{e = \int \vec{E}\cdot\vec{dL} = -\int \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds}}$$

This is the Integral form of Maxwell's eqn, derived from Faraday's law.

Statement :- " The total electromotive force (emf) induced in a closed path is equal to the -ve surface integral of the rate of change of magnetic flux density over an entire surface bounded by the same closed path."

Using Stoke's theorem, converting line integral into surface integral:

$$\int (\nabla \times \vec{E}) \cdot \vec{ds} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

$$\Rightarrow \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

This is the Maxwell's eqn. derived from Faraday's law in point form or differential form.

2) Ampere's Circuital law :-

According to Ampere's circuital law,

$$\oint \vec{H} \cdot \vec{dL} = I$$

Also $I = \int \vec{J} \cdot \vec{ds}$ where $\vec{J} = \vec{J_c} + \vec{J_D}$

$$\therefore \boxed{\oint \vec{H} \cdot \vec{dL} = \int (\vec{J_c} + \vec{J_D}) \cdot \vec{ds}}$$

$$\text{or} \quad \boxed{\oint \vec{H} \cdot \vec{dL} = \int \left(J_c + \frac{\partial \vec{D}}{\partial t}\right) \cdot \vec{ds}}$$

This is the integral form of M.E derived from Ampere's circuital law,

Statement: " The total magneto motive force around any closed path is equal to the surface integral of conduction & displacement current densities over the entire surface bounded by the same closed path."

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Using Stoke's theorem,

$$\oint \vec{H} \cdot d\vec{L} = \int (\nabla \times \vec{H}) \cdot d\vec{s}$$

$$\therefore \boxed{\nabla \times \vec{H} = \vec{J_c} + \frac{\partial \vec{D}}{\partial t}}$$

This is the M.E derived from Ampere's law in point form or differential form.

3) Gauss's law (for static field):-

According to Gauss's law, total flux leaving (passing through) a closed surface is equal to the net charge enclosed by the surface

$$\psi = Q = \oint \vec{D} \cdot d\vec{s}$$

Also $Q = \int \rho_v \, dV$

$$\therefore \boxed{\oint \vec{D} \cdot d\vec{s} = \int \rho_v \, dV}$$ Integral form of M.E

Applying divergence theorem,

$$\oint \vec{D} \cdot d\vec{s} = \int (\nabla \cdot \vec{D}) \, dV$$

$$\therefore \int (\nabla \cdot \vec{D}) \, dV = \int \rho_v \, dV$$

$$\Rightarrow \boxed{\nabla \cdot \vec{D} = \rho_v}$$ Differential form or point form of M.E.

4) Gauss's law (for magnetic field):-

Statement:- The surface integral of magnetic flux density over a closed surface is always equal to zero.

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$$\boxed{\oint_{closed} \vec{B} \cdot \vec{ds} = 0}$$ Integral form of M.E.

Applying Divergence theorem.

i.e $\oint_{closed} \vec{B} \cdot \vec{ds} = \int (\nabla \cdot \vec{B})\, dv$

$\Rightarrow \quad 0 = \int (\nabla \cdot \vec{B})\, dv$

But $dv \neq 0$ $\therefore$ $\boxed{\nabla \cdot \vec{B} = 0}$ Point form of M.E

Differential form	Integral form	Significance
1) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\int \vec{E} \cdot \vec{dL} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$	Faraday's law
2) $\nabla \times \vec{H} = \vec{J_c} + \frac{\partial \vec{D}}{\partial t}$	$\int \vec{H} \cdot \vec{dL} = \int \left(J_c + \frac{\partial D}{\partial t}\right) \cdot \vec{ds}$	Ampere's circuital law.
3) $\nabla \cdot \vec{D} = \rho_v$	$\int \vec{D} \cdot \vec{ds} = \int \rho_v\, dv$	Gauss's law
4) $\nabla \cdot \vec{B} = 0$	$\oint \vec{B} \cdot \vec{ds} = 0$	Gauss's law for mag field.

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Uniform Plane Waves

The waves are the means of transporting energy or information from source to destination. The waves consisting of electric & magnetic fields are called EM waves.

* **General Wave Equation :-**

In general, the wave equations can be obtained by relating space & time variations of the electric field & magnetic fields.

Consider the Maxwell's Equation

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \text{——} (1)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \nabla \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{——} (2)$$

$$\nabla \cdot \vec{H} = 0 \quad \text{——} (3)$$

$$\& \ \nabla \cdot \vec{E} = 0 \quad \text{——} (4)$$

W.K.T
$\nabla \cdot \vec{D} = \rho_v$
but $\rho_v = 0$ for time varying fields
$\nabla \cdot \vec{D} = 0$
or $\nabla \cdot E = 0$

Taking curl on both the sides of Eqn (1) we get

$$\nabla \times \nabla \times \vec{E} = -\mu \left(\nabla \times \frac{\partial \vec{H}}{\partial t} \right)$$

∇ operator & $\frac{\partial}{\partial t}$ operator can be interchanged (∵ both are differential operators)

$$\therefore \ \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H})$$

Using Eqn (2)

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \left[\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \right]$$

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$$\nabla \times \nabla \times \vec{E} = -\mu\sigma \frac{\partial E}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

W.K.T $\nabla \times \nabla \times \vec{E} = \nabla.(\nabla.\vec{E}) - \nabla^2\vec{E}$

But from eqn (4) $\nabla.\vec{E} = 0$

$$\therefore \quad -\nabla^2\vec{E} = -\mu\sigma \frac{\partial E}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2\vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}} \quad \text{--- (5)}$$

'ully

$$\boxed{\nabla^2\vec{D} = \mu\sigma \frac{\partial \vec{D}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{D}}{\partial t^2}} \quad \text{--- (6)}$$

Similar Eqns can be written for magnetic field $\vec{H}$ & for $\vec{B}$ as,

$$\nabla^2\vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} \quad \text{--- (7)}$$

$$\& \quad \nabla^2\vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{--- (8)}$$

1* Uniform Plane Waves in free space:-

Consider an EM wave propagating through free space ($\sigma = 0$). Let the wave be propagating in z direction with E13 electric field in the x-direction & magnetic field in the y-direction. These fields vary with time as wave propagates in the free space.

$\therefore$ E_x & H_y are functions of "z" & "t" only

Also $\vec{E_x}$ & $\vec{H_y}$ are mutually $\perp^r$ to each other & also they are $\perp^r$ to the direction of wave propagation. Hence EM waves are also called Transverse EM waves (TEM).

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Consider wave Eqns for $\vec{E}$ & $\vec{H}$

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 E}{\partial t^2} \quad \text{—— (1)}$$

$$\nabla^2 \vec{H} = \mu\sigma \frac{\partial H}{\partial t} + \mu\epsilon \frac{\partial^2 H}{\partial t^2} \quad \text{—— (2)}$$

For free space $(\sigma = 0, \mu = \mu_0, \epsilon = \epsilon_0)$

$\therefore$ Eqn (1) becomes, $\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$ —— (3)

Since E is in x- direction & varies w.r.t z- direction

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \quad \text{—— (4)}$$

Let $E_x = E_m e^{j\omega t}$; $E_m \rightarrow$ amplitude of the electric field

$\omega \rightarrow$ angular frequency.

Diff w.r.t 't'

$$\frac{\partial E_x}{\partial t} = E_m \times j\omega\, e^{j\omega t}$$

Diff w.r.t 't'

$$\frac{\partial^2 E_x}{\partial t^2} = E_m\, j\omega \times j\omega\, e^{j\omega t} = -\omega^2 E_x$$

$j^2 = -1$

Substituting in eqn (4),

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \times -\omega^2 E_x$$

$$\Rightarrow \frac{\partial^2 E_x}{\partial z^2} + \omega^2 \mu_0 \epsilon_0 E_x = 0 \quad \text{—— (5)}$$

Let $\frac{\partial^2}{\partial z^2} = D^2$ $\quad \therefore D^2 E_x + \omega^2 \mu_0 \epsilon_0 E_x = 0$

$$E_x (D^2 + \omega^2 \mu_0 \epsilon_0) = 0$$

as $E_x \neq 0$ $\quad D^2 + \omega^2 \mu_0 \epsilon_0 = 0$

or $D^2 = -\omega^2 \mu_0 \epsilon_0$

$$D = \pm\sqrt{-\omega^2 \mu_0 \epsilon_0}$$

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$\sqrt{-1} = j$

$$D = \pm j\omega\sqrt{\mu_0 \epsilon_0} = \pm j\beta$$

where $\beta = \omega\sqrt{\mu_0\epsilon_0}$ called phase shift constant

$\therefore$ the solution of Eqn (5) is,

$$E_x = K_1 e^{-j\beta z} + K_2 e^{+j\beta z}$$ where K_1 & K_2 are constants.

Assuming $K_1 = K_2 = E_m e^{j\omega t}$

$$\therefore E_x = E_m e^{j\omega t} e^{-j\beta z} + E_m e^{j\omega t} \cdot e^{j\beta z}$$

$$\boxed{E_x = E_m e^{j(\omega t - \beta z)} + E_m e^{j(\omega t + \beta z)}} \quad \text{--- (6)}$$

Taking real part of the above Eqn.

$$\boxed{E_x = E_m \cos(\omega t - \beta z) + E_m \cos(\omega t + \beta z)}$$

It consists of two components; one in forward direction & other in backward direction.

$||^{ly}$ $\boxed{H_y = H_m \cos(\omega t - \beta z) + H_m \cos(\omega t + \beta z)}$

Thus E_x & H_y are only functions of Time & space (direction of travel).

* **Expression for intrinsic impedance (η) in free space :-**

** Consider the Maxwell's Eqn derived from Faraday's law,

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\begin{vmatrix} \overset{+}{\hat{a}_x} & \overset{-}{\hat{a}_y} & \overset{+}{\hat{a}_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\mu \frac{\partial H_y}{\partial t}$$

Assuming wave is propagating in z direction. E_x & H_y are present.

$$-\hat{a}_y\left[0-\frac{\partial E_x}{\partial z}\right] = -\mu\frac{\partial H_y}{\partial t}\hat{a}_y$$

$$\frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t}$$

Since the wave is propagating in free space. The expressions for E_x & H_y are

$$E_x = E_m\cos(\omega t-\beta z) \quad \& \quad H_y = H_m\cos(\omega t-\beta z)$$

$$\frac{\partial}{\partial z}\left[E_m\cos(\omega t-\beta z)\right] = -\mu\frac{\partial}{\partial t}\left[H_m\cos(\omega t-\beta z)\right]$$

$$E_m\times -\sin(\omega t-\beta z)\times -\beta = -\mu H_m\times -\sin(\omega t-\beta z)\times\omega$$

$$\Rightarrow \beta E_m \cancel{\sin(\omega t-\beta z)} = \mu H_m\omega \cancel{\sin(\omega t-\beta z)}$$

$$\Rightarrow \beta E_m = \mu H_m\omega$$

$$\Rightarrow \frac{E_m}{H_m} = \frac{\mu\omega}{\beta}$$

For free space $\beta = \omega\sqrt{\mu_0\epsilon_0}$

$$\therefore \frac{E_m}{H_m} = \frac{\mu_0\cancel{\omega}}{\cancel{\omega}\sqrt{\mu_0\epsilon_0}} = \frac{\sqrt{\mu_0}\sqrt{\mu_0}}{\sqrt{\mu_0\epsilon_0}}$$

$$\boxed{\eta = \sqrt{\frac{\mu_0}{\epsilon_0}}}$$ where η is called intrinsic impedance of the medium

for free space, $\mu_0 = 4\pi\times10^{-7}$ H/m

$\epsilon_0 = 8.854\times10^{-12}$ F/m

$$\therefore \eta = \sqrt{\frac{4\pi\times10^{-7}}{8.854\times10^{-12}}} = 377\,\Omega \text{ or } 120\pi\,\Omega$$

Thus for free space $\boxed{\eta = 377\,\Omega}$

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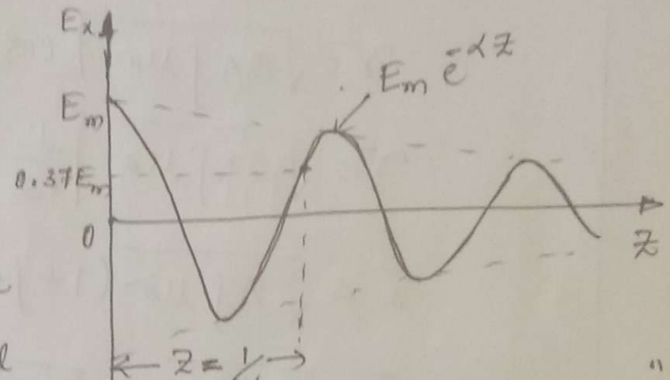
Consider E_x expression,

$$E_x = \underbrace{E_m e^{-\alpha z}}_{amp} \underbrace{\cos(\omega t - \beta z)}_{phase}$$

Let us consider only amplitude term of E_x

$$E_x = E_m e^{-\alpha z}$$

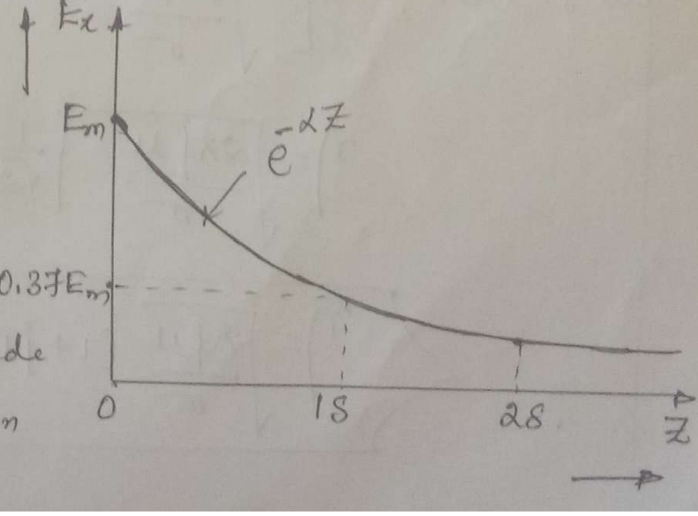
At $z=0$, amplitude of the component E_x is E_m; while at $z=d$, amplitude is $E_m e^{-\alpha z}$. If $d = \frac{1}{\alpha}$, then the factor becomes $E_x = E_m e^{-1} = 0.368 E_m$. So over a distance of $z = 1/\alpha$, the amplitude of the wave decreases to approximately 37% of its original value.



"The distance through which the amplitude of the travelling wave decrease to 37% of the original amplitude is called Skin depth or depth of penetration (δ)."

$$\boxed{\text{Skin depth } (\delta) = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}}$$

From the expression it is clear that, $\delta \propto \frac{1}{\sqrt{f}}$ (i.e, inversly αnal to the square root of freq. So for μw [micro wave] frequencies the skin depth is very small.



From the graph, it is clear that at 1δ distance amplitude reduces to 37% of its original value.

For a good conductor, amplitude reduces to almost zero within 2δ or 3δ distance.

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For good conductor ($\sigma >> \omega\epsilon$)
The propogation constant is given by,

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

As $\sigma >> \omega\epsilon$ (omitting 2nd term inside the bracket)

$$\gamma = \sqrt{j\omega\mu\sigma}$$

$$\gamma = \sqrt{\omega\mu\sigma}\sqrt{j} = \sqrt{\omega\mu\sigma}\sqrt{\angle 90^\circ}$$

$j = \angle 90^\circ$

$\sqrt{\text{angle}} = \frac{1}{2}\text{ angle}$

$$\Rightarrow \quad \gamma = \sqrt{\omega\mu\sigma}\,\angle 45^\circ$$

$$\gamma = \sqrt{2\pi f\mu\sigma}\left[\cos 45^\circ + j\sin 45^\circ\right]$$

$$\gamma = \sqrt{2\pi f\mu\sigma}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$\gamma = \sqrt{\pi f\mu\sigma}\,(1 + j1)$$

$$\gamma = \sqrt{\pi f\mu\sigma} + j\sqrt{\pi f\mu\sigma}$$ comparing this with $\gamma = \alpha + j\beta$

$$\boxed{\alpha = \sqrt{\pi f\mu\sigma}}\ \text{Np/m} \quad \& \quad \boxed{\beta = \sqrt{\pi f\mu\sigma}}\ \text{rad/m}$$

The intrinsic impedance of a good conductor is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

as $\sigma >> \omega\epsilon$, $\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\sqrt{j}$

$$\eta = \sqrt{\frac{2\pi f\mu}{\sigma}}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$\boxed{\eta = \sqrt{\frac{\pi f\mu}{\sigma}}\,(1 + j1)}$$

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* Loss tangent & its importance :-

The conduction current density is given by,

$$\vec{J_c} = \sigma \vec{E} \quad \text{—— (1)}$$

& Displacement current density is,

$$\vec{J_D} = \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

But $\frac{\partial \vec{E}}{\partial t} = j\omega E$

$$\therefore \vec{J_D} = j\omega\epsilon\vec{E} \quad \text{—— (2)}$$

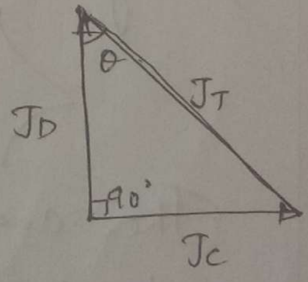
The ratio of the conduction current density to the displacement current density is given by

$$\frac{\vec{J_c}}{\vec{J_D}} = \frac{\sigma \cancel{\vec{E}}}{j\omega\epsilon\cancel{\vec{E}}}$$

$$\Rightarrow \boxed{\frac{J_c}{J_D} = \frac{\sigma}{j\omega\epsilon}}$$

From the Eqn, it is clear that displacement current density leads conduction current density by 90°.

If θ is the angle between total current density & the displacement current density.



$$\tan\theta = \frac{J_c}{J_D} = \frac{\sigma}{\omega\epsilon}$$

$$\tan\theta = \frac{\sigma}{\omega\epsilon}$$

The term $\frac{\sigma}{\omega\epsilon}$ is called <u>loss tangent</u> & θ is called <u>loss angle</u>. When $\sigma >> \omega\epsilon$, the loss tangent is very high, thus medium is said to be good conductor. When $\sigma << \omega\epsilon$ the loss tangent is also small, thus medium is said to be good dielectric.

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Poynting Vector and Poynting Theorem :-

At any point in an EM field, the power density or power per unit area is given by

$$\vec{P} = \vec{E} \times \vec{H} \quad W/m^2$$

where $\vec{P}$ is called Poynting vector. The direction of $\vec{P}$ indicates instantaneous power flow at that point.

Suppose $\vec{E} = E_x \, \hat{a}_x$ & $\vec{H} = H_y \, \hat{a}_y$

$$\vec{P} = \vec{E} \times \vec{H}$$

$$\vec{P} = E_x \, \hat{a}_x \times H_y \, \hat{a}_y$$

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$

$$\vec{P} = E_x H_y \, \hat{a}_z$$

$$\boxed{\vec{P} = P_z \, \hat{a}_z}$$

The power flow is in z-direction. i.e, in the direction of wave propogation.

Average power density [P_{avg}] :-

The average power density for free space & perfect dielec[tric] is given by,

$$P_{avg} = \frac{E_m^2}{2\eta}$$

The average power density for lossy media & for good conductor is,

$$P_{avg} = \frac{E_m^2}{2\eta} e^{-2\alpha z} \cos(\theta_\eta)$$

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** **Integral & point forms of Poynting theorem :-**

Statement :- "The net power flowing out of a given volume V is equal to the time rate of decrease in the energy stored within volume minus the ohmic power dissipated."

Consider the Maxwell's Eqn

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \text{—— (1)}$$

&

$$\nabla \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{—— (2)}$$

Taking $\vec{E}.$ on both side of eqn (2)

$$\vec{E}.(\nabla \times \vec{H}) = \vec{E}.(\sigma \vec{E}) + \vec{E}.\left(\epsilon \frac{\partial \vec{E}}{\partial t}\right) \quad \text{—— (3)}$$

Using vector identity,

$$\vec{E}.(\nabla \times \vec{H}) = \vec{H}.(\nabla \times \vec{E}) - \nabla.(\vec{E} \times \vec{H})$$

$\therefore$ Eqn (3) becomes,

$$\vec{H}.(\nabla \times \vec{E}) - \nabla.(\vec{E} \times \vec{H}) = \sigma E^2 + \epsilon\left(\vec{E}.\frac{\partial \vec{E}}{\partial t}\right)$$

Substituting $\nabla \times \vec{E}$ & $\vec{P} = \vec{E} \times \vec{H}$ in the above Eqn.
eqn (1)

$$\vec{H}. -\mu \frac{\partial \vec{H}}{\partial t} - \nabla.\vec{P} = \sigma E^2 + \epsilon\left(\vec{E}.\frac{\partial \vec{E}}{\partial t}\right)$$

$$-\mu\, \vec{H}.\frac{\partial \vec{H}}{\partial t} - \nabla.\vec{P} = \sigma E^2 + \epsilon\left(\vec{E}.\frac{\partial \vec{E}}{\partial t}\right)$$

$$-\nabla.\vec{P} = \sigma E^2 + \epsilon\left(\vec{E}.\frac{\partial E}{\partial t}\right) + \mu\left(H.\frac{\partial \vec{H}}{\partial t}\right)$$

$$-\nabla.\vec{P} = \sigma E^2 + \frac{1}{2}\epsilon \frac{\partial E^2}{\partial t} + \frac{1}{2}\mu \frac{\partial H^2}{\partial t}$$

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$$\boxed{-\nabla\cdot\vec{P} = \underbrace{\sigma E^2}_{\text{ohmic loss}} + \frac{\partial}{\partial t}\underbrace{\left[\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right]}_{\text{Energy stored in terms of } \mu \,\&\, \epsilon}}$$

The above Eqn represents poynting theorem in point form.

Taking volume integral on both side,

$$-\int_v \nabla\cdot\vec{P}\,dv = \int_v \sigma E^2 dv + \int_v \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right] dv$$

applying divergence to LHS

$$\boxed{\oint \vec{P}\cdot\vec{ds} = -\int_v \sigma E^2 dv - \int_v \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right] dv}$$

The above Eqn represents poynting theorem in integral form.

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